Math 404/541 Homework 3

- Due Friday Nov 29 at 12:00pm.
- Homework should be submitted using Canvas.
- **Collaboration Policy**: You are welcome (and encouraged) to work on the homework in groups. However, each student must write up the homework on their own, and must use their own wording (i.e. don't jusy copy the solutions from your friend). If you do collaborate with others, please list the name of your collaborators at the top of the homework.
- Homework should be typeset in LaTeX. If you are unfamiliar with LaTeX, "The Not So Short Introduction to LaTeX" is a good place to start. This guide can be found at http://tug.ctan.org/info/lshort/english/lshort.pdf . You can also download the .tex source file for this homework and take a look at that.
- Each homework problem should be correct as stated. Occasionally, however, I might screw something up and give you an impossible homework problem. If you believe a problem is incorrect, please email me. If you are right, the first person to point out an error will get +1 on that homework, and I will post an updated version.

1. Recall that

$$\|f\|_{\mathrm{BMO}(\mathbb{R}^n)} = \sup_B \frac{1}{|B|}\int_B |f - f_B|,$$

where $f_B = \frac{1}{|B|}\int_B f$, and the supremum is taken over all balls $B \subset \mathbb{R}^n$.

a) Prove that $\|f+g\|_{\mathrm{BMO}(\mathbb{R}^n)} \leq \|f\|_{\mathrm{BMO}(\mathbb{R}^n)} + \|g\|_{\mathrm{BMO}(\mathbb{R}^n)}$.

b) Prove that for each $1 \leq p < \infty$,

$$\|f\|_{\mathrm{BMO}(\mathbb{R}^n)} \sim \sup_B \left(\frac{1}{|B|}\int_B |f - f_B|^p\right)^{1/p},$$

and

$$\|f\|_{\mathrm{BMO}(\mathbb{R}^n)} \sim \sup_B \inf_{c\in\mathbb{C}} \left(\frac{1}{|B|}\int_B |f - c|^p\right)^{1/p},$$

where the implicit constant can depend on n and p (you should also prove this for $p = 1$, since we did not prove this in class)

c) Prove that $\|\log|x|\|_{\mathrm{BMO}(\mathbb{R}^n)} < \infty$.

2. Let $1 \le j \le n$. If $f\colon \mathbb{R}^n \to \mathbb{C}$ and $x \in \mathbb{R}^n$, define

$$R_j f(x) = c_n \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n \setminus B(x,\varepsilon)} \frac{(y_j - x_j) f(y)}{|x-y|^{n+1}} dy.$$

If the constant c_n is chosen appropriately, then whenever $f \in \mathcal{S}(\mathbb{R}^n)$ we have

$$\widehat{R_j f}(\xi) = -i \frac{\xi_j}{|\xi|} \hat{f}(\xi).$$

(You do not need to prove this).

Prove that R_j is a Calderón-Zygmund operator.

3. a) Let $\Omega \subset \mathbb{R}^n$ be a convex polytope (i.e. a convex set given by the intersection of finitely many half-spaces). Define the linear operator T_Ω as follows: If $f \in \mathcal{S}(\mathbb{R}^n)$, then $\widehat{T_\Omega f}(\xi) = \chi_\Omega(\xi)\hat{f}$. Prove that for each $1 < p < \infty$, T_Ω extends to a bounded linear operator from $L^p(\mathbb{R}^n)$ to $L^p(\mathbb{R}^n)$.

Hint: First prove the following result about operators: Suppose that $S\colon L^p(\mathbb{R}) \to L^p(\mathbb{R})$ is bounded, and define the operator T by $Tf(x_1,\ldots,x_n) = Sg_{x_1,\ldots,x_{n-1}}(x_n)$, were $g_{x_1,\ldots,x_{n-1}}(x_n) = f(x_1,\ldots,x_n)$. Then $T\colon L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is bounded.

b) Let $\Omega \subset \mathbb{R}^2$ be a polygon (not necessarily convex). Prove that for each $1 < p < \infty$, T_Ω extends to a bounded linear operator from $L^p(\mathbb{R}^2)$ to $L^p(\mathbb{R}^2)$.