Math 404/541 Homework 2

- Due Friday Nov 8 at 12:00pm.
- Homework should be submitted using Canvas.
- **Collaboration Policy**: You are welcome (and encouraged) to work on the homework in groups. However, each student must write up the homework on their own, and must use their own wording (i.e. don't jusy copy the solutions from your friend). If you do collaborate with others, please list the name of your collaborators at the top of the homework.
- Homework should be typeset in LaTeX. If you are unfamiliar with LaTeX, "The Not So Short Introduction to LaTeX" is a good place to start. This guide can be found at http://tug.ctan.org/info/lshort/english/lshort.pdf . You can also download the .tex source file for this homework and take a look at that.
- Each homework problem should be correct as stated. Occasionally, however, I might screw something up and give you an impossible homework problem. If you believe a problem is incorrect, please email me. If you are right, the first person to point out an error will get +1 on that homework, and I will post an updated version.

In the problems below, $\mathcal{S}(\mathbb{R}^n)$ refers to the set of Schwartz functions $\phi\colon \mathbb{R}^n \to \mathbb{C}$. If the dimension n is obvious from context (or doesn't matter), then we will use $\mathcal{S}$ instead of $\mathcal{S}(\mathbb{R}^n)$.

1. Let $f \in L^2(\mathbb{R})$, and suppose that $\operatorname{supp}(f) \subset [-R, R]$ for some $R > 0$. For each $z \in \mathbb{C}$, define

$$\mathcal{G}f(z) = \int e^{2\pi i x z} f(x) dx.$$

a) Prove that for each $z \in \mathbb{C}$, the expression $\mathcal{G}f(z)$ is well-defined (i.e. the integral converges), and satisfies the inequality

$$|\mathcal{G}f(z)| \lesssim e^{2\pi R|z|},$$

where the implicit constant might depend in f, but is not allowed to depend on z.

b) Define $u(x, y) = \operatorname{Re} \mathcal{G}f(x + iy)$ and $v(x, y) = \operatorname{Im} \mathcal{G}f(x + iy)$. Prove that

$$\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y}, \\ \frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x}. \end{aligned} \tag{1}$$

(these equations should look familiar!)

c) Let $f \in L^2(\mathbb{R})$, and suppose that f has compact support. Prove that $\hat{f}$ is analytic.

d) Let $f \in L^2(\mathbb{R})$, and suppose that f and $\hat{f}$ have compact support. Prove that $f = 0$.

2. Define $C_0(\mathbb{R}^n)$ to be the set of continuous functions $g\colon \mathbb{R}^n \to \mathbb{C}$ so that $|g(x)| \to 0$ as $|x| \to \infty$. Prove that if $f \in L^1(\mathbb{R}^n)$, then there is a function $g \in C_0(\mathbb{R}^n)$ so that $\hat{g} = f$ in the sense of distributions (i.e. $\hat{u}_g(\phi) = u_f(\phi)$ for all $\phi \in \mathcal{S}$).

3. If $f, g\colon \mathbb{R}^m \to \mathbb{C}$, define the tensor product $f \otimes g\colon \mathbb{R}^{2m} \to \mathbb{C}$ by $f \otimes g(x, y) = f(x)g(y)$. Fix $n \geq 1$. Define $\mathcal{A}$ to be the set of all functions $\mathbb{R}^n \to \mathbb{C}$ of the form $f_1(x_1) \otimes f_2(x_2) \otimes \cdots \otimes f_n(x_n)$, where $f_1, \ldots, f_n \in \mathcal{S}(\mathbb{R})$. Define $\mathcal{B}_n$ to be the set of all (finite) complex linear combinations of functions from $\mathcal{A}_n$, i.e. sums of the form $c_1 g_1(x) + \ldots + c_k g_k(x)$, where $c_1, \ldots, c_k \in \mathbb{C}$ and $g_1, \ldots, g_k \in \mathcal{A}_n$.

a) Prove that $\mathcal{B}_n \subset \mathcal{S}(\mathbb{R}^n)$.

b) Prove that $\mathcal{B}_n$ is dense in $\mathcal{S}(\mathbb{R}^n)$.

c) Prove that for every $1 \leq p < \infty$, $\mathcal{B}_n$ is dense in $L^p(\mathbb{R}^n)$.

4. Prove that there cannot exist a bound of the form

$$\|\hat{f}\|_q \leq C_{p,q} \|f\|_p \quad \text{for all } f \in L^p(\mathbb{R}^n)$$

unless $1 \leq p \leq 2$ and $q = p'$.

5. Let $u \in \mathcal{S}'$ be a tempered distribution. If $f \in \mathcal{S}$, we define the function $f * u\colon \mathbb{R}^n \to \mathbb{C}$ by $f * u(z) = u(S_z f)$, where $S_z f(x) = f(z - x)$. Also recall that if $f \in \mathcal{S}$, then we can define $fu(\phi) = u(f\phi)$.

a) Prove that if $u \in \mathcal{S}'$ and $f \in \mathcal{S}$, then $\widehat{fu} = \hat{f} * \hat{u}$ and $\widehat{u * f} = \hat{u}\hat{f}$.

b) Prove that if $u \in \mathcal{S}'$ and $f \in \mathcal{S}$, then $f * u(x) \in C^\infty(\mathbb{R}^n)$.

c) Let $u \in \mathcal{S}'$. Suppose that u is compactly supported, in the sense that there exists a compact set $K \subset \mathbb{R}^n$ so that $u(\phi) = 0$ for all $\phi \in \mathcal{S}$ with $\operatorname{supp}(\phi) \cap K = \emptyset$. Prove that $\hat{u}$ is smooth, in the sense that there exists a function $g \in C^\infty(\mathbb{R}^n)$ so that $\hat{u} = u_g$.