1. a) Prove that $Mf \in L^1(\mathbb{R}^n)$ if and only if $f = 0$ a.e.

b) If $f \in L^1(\mathbb{R}^n)$, it need not even be true that $Mf \in L^1_{\text{loc}}(\mathbb{R}^n)$. Find an example of a function $f \in L^1(\mathbb{R}^n)$ so that $Mf \not\in L^1_{\text{loc}}(\mathbb{R}^n)$, and show that your example is correct. Hint: the statement of part (c) might be useful for your search.

c) Let $f: \mathbb{R}^n \to \mathbb{C}$ and suppose that
\[
\int |f(x)| \log(2 + |f(x)|) dx < \infty.
\]
Prove that $Mf \in L^1_{\text{loc}}(\mathbb{R}^n)$.

Hint: If $K$ is a compact set, then
\[
\int_K Mf(x) dx \leq \int_{K \cap \{|Mf| < 1\}} Mf(x) dx + \int_{Mf \geq 1} Mf(x) dx \leq |K| + \int_{Mf \geq 1} Mf(x) dx.
\]

Solution. a) We prove the only if part. Assume w.l.o.g that $\int_{|x| < 1} |f| = c > 0$, otherwise we do a translation. It suffices to prove $\int_{|x| > 1} Mf = \infty$. Given $|x| > 1$, the key is to notice that $B(x, 2|x|) \supset B(0, |x|) \supset B(0, 1)$. This gives $Mf(x) \gtrsim |x|^{-n}$ for $|x| > 1$.

b) Consider the function
\[
f(x) = \sum_{k=1}^{\infty} \frac{2^{kn}}{k^2} 1_{[2^{-k-1}, 2^{-k})}(|x|).
\]
Then $f \in L^1(\mathbb{R}^n)$. For $0 < |x| \leq 1/2$, use $B(x, 2|x|) \supset B(0, |x|)$ to show that $Mf(x) \gtrsim |x|^{-n} / (-\log |x|)$.

c) We use the hint. By Tonelli’s theorem we can write
\[
\int_{Mf \geq 1} Mf(x) dx = \int_1^\infty \mu\{x : Mf(x) > t\} dt.
\]
Split $f = f_1 + f_2$ where $f_1 = f1_{|f| \leq t/2}$ and $f_2 = f1_{|f| > t/2}$. Then $Mf \leq t/2 + Mf_2$, so
\[
\mu\{x : Mf(x) > t\} \leq \mu\{x : Mf_2(x) > t/2\} \lesssim t^{-1} \|f_2\|_1.
\]
Thus
\[ \int_1^\infty \mu\{x : Mf(x) > t\} \, dt \lesssim \int_1^\infty t^{-1} \int_{|f| > t/2} |f(x)| \, dx \, dt \]
(by Tonelli’s) = \[ \int_{|f| > t/2} |f(x)| \log(2|f(x)|) \, dx \]
\[ \lesssim \int |f(x)| \log(2 + |f(x)|) \, dx. \]

2. Let \( \phi : \mathbb{R}^n \to [0, \infty] \) be radial and non-increasing, i.e. \( \phi(x) = \phi(y) \) whenever \( |x| = |y| \), and \( \phi(x) \leq \phi(y) \) whenever \( |x| \geq |y| \).

Prove that if \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \), then \( |\phi \ast f(x)| \leq \|\phi\|_1 Mf(x) \) for all \( x \in \mathbb{R}^n \) with \( Mf(x) < \infty \).

\textbf{Solution.} Step 1: Prove this for \( \phi(x) = 1_{|x| \leq R} \) where \( R > 0 \).

Step 2: Let \( \mathcal{D} \) be the collection of all radial functions \( \phi : \mathbb{R}^n \to [0, \infty] \) of the form
\[ \phi(x) = \sum_{i=1}^N s_i 1_{b_{i-1} \leq |x| < b_i}, \]
where \( 0 \leq s_N \leq s_{N-1} \leq \cdots \leq s_1 < \infty \) and \( 0 = b_1 < b_2 < \cdots < b_N \). We then prove the required inequality for all \( \phi \in \mathcal{D} \).

\textit{Hint:} For \( \phi \in \mathcal{D} \), we can write
\[ \phi(x) = \sum_{i=1}^N c_i 1_{|x| < b_i}, \]
where \( c_i = s_i - s_{i+1} \geq 0 \) for \( 1 \leq i \leq N \) (let \( s_{N+1} = 0 \)).

Step 3: Given \( \phi : \mathbb{R}^n \to [0, \infty] \) be radial and radially decreasing. Define \( \phi^0 : [0, \infty) \to [0, \infty] \) by \( \phi^0(r) = \phi(r, 0, 0, \ldots, 0) \). For each \( j \), we define
\[ \phi_j(x) = \sum_{i=1}^{4^j} \phi^0(i2^{-j}) 1_{i2^{-j} \leq |x| < i2^{-j}}. \]
Show that \( \phi_j(x) \not\nearrow \phi(x) \) for all continuity points \( x \) of \( \phi \).

Step 4: Show that for any decreasing \( \phi^0 : [0, \infty) \to [0, \infty] \), \( \phi^0 \) has at most countably many discontinuity points. \textit{Hint:} A monotone function can only have jump discontinuities. Fix a compact \( K \subseteq (0, \infty) \) and \( k \in \mathbb{N} \). Show that the collection \( C_{k,K} \) of all points \( x \in K \) where the jump is greater than \( k^{-1} \) is finite. Note that the set of all discontinuity points of \( \phi^0 \) on \( [0, \infty) \) can be written as a countable union of sets of the form \( C_{k,K} \).

Step 5: Use Step 4 to conclude that \( \phi_j(x) \not\nearrow \phi(x) \) a.e. Apply Step 2 to each \( \phi_j \) and then use monotone convergence theorem.
\[ \square \]
If \( f \in C_c(\mathbb{R}^n) \) and \( r > 0 \), define
\[
A_r f(x) = \int_{S^{n-1}} f(x + ry) d\sigma(y),
\]
where \( S^{n-1} \) is the \( d-1 \) dimensional unit sphere in \( \mathbb{R}^n \), and \( \sigma \) is normalized surface measure on \( S^{n-1} \) (i.e. \( \sigma(S^{n-1}) = 1 \)). Define
\[
M_S f(x) = \sup_{r > 0} A_r |f|(x).
\]

\( M_S \) is called the spherical maximal operator. Stein proved that if \( n \geq 3 \) and \( n/(n-1) < p \leq \infty \), then there is a constant \( C_{n,p} \) so that
\[
\|M_S f\|_p \leq C_{n,p} \|f\|_p \quad \text{for all } f \in L^p(\mathbb{R}^n) \cap C_c(\mathbb{R}^n).
\]
(1)

Later, Bourgain proved that (1) holds when \( n = 2 \) and \( 2 < p \leq \infty \).

Using this result, prove that if \( f \in L^p_{\text{loc}}(\mathbb{R}^n) \) for the same range of \( p \) as above, then
\[
\lim_{r \searrow 0} A_r f(x) = f(x) \quad \text{a.e.}
\]

**Bonus.** Previously, this problem asked you to prove that
\[
\lim_{r \searrow 0} A_r f(x) = f(x) \quad \text{a.e.}
\]
for all \( f \in L^1_{\text{loc}}(\mathbb{R}^n) \). Give a counter-example showing that this is not true.

**Solution.** W.l.o.g assume \( f \) is supported on a compact set, and \( p < \infty \). (Why?) Then \( f \in L^p(\mathbb{R}^n) \), so for any \( \epsilon > 0 \) we may find a continuous \( g \in L^p(\mathbb{R}^n) \) such that \( \|f - g\|_p < \epsilon \). Then split
\[
\limsup_{r \to 0^+} |A_r f(x) - f(x)|
\]
\[
\leq \limsup_{r \to 0^+} |A_r f(x) - A_r g(x)| + \limsup_{r \to 0^+} |A_r g(x) - g(x)| + |g(x) - f(x)|
\]
\[
= \limsup_{r \to 0^+} |A_r (f - g)(x)| + |g(x) - f(x)|
\]
\[
\leq M(f - g)(x) + |g(x) - f(x)|.
\]

Thus for each \( t > 0 \),
\[
\mu\{x : \limsup_{r \to 0^+} |A_r f(x) - f(x)| > t\}
\]
\[
\leq \mu\{x : M(f - g)(x) > t/2\} + \mu\{x : |f(x) - g(x)| > t/2\}
\]
\[
\lesssim t^{-1} \|f - g\|_p^p < t^{-1} \epsilon^p.
\]

Since this holds for any \( \epsilon > 0 \), we have \( \mu\{x : \limsup_{r \to 0^+} |A_r f(x) - f(x)| > t\} = 0 \). Since this holds for any \( t > 0 \), we have \( A_r f(x) \to f(x) \) a.e. \( \square \)

For the bonus question, refer to the appendix.
4. In lecture we discussed the vector space \( L^p(\mathbb{R}^n) + L^p(\mathbb{R}^n) \) consisting of all equivalence classes of measurable functions that can be written as \( g + h \), where \( g \in L^p(\mathbb{R}^n) \) and \( h \in L^p(\mathbb{R}^n) \). If \( f \in L^p(\mathbb{R}^n) + L^p(\mathbb{R}^n) \), define

\[
\|f\|_{L^p + L^p} = \inf\{\|g\|_{L^p} + \|h\|_{L^p} : f = g + h\}.
\]

a) Prove that with this definition, \( L^p + L^p \) is a normed vector space, i.e. prove that \( \|\cdot\|_{L^p + L^p} \) is indeed a norm.

b) Is this normed vector space complete? Prove that your answer is correct.

**Solution.**

a) We abbreviate \( \|\cdot\|_{L^p + L^p} \) as \( \|\cdot\| \).

- It is trivial that \( \|f\| \geq 0 \) and that \( \|0\| = 0 \). Conversely, let \( f \) so that \( \|f\| = 0 \). Suppose, towards contradiction, that \( f \) is not 0 a.e. Then there exists a set \( A \subset \mathbb{R}^n \) of positive measure and a number \( t > 0 \) so that \( |f(x)| > t \) for all \( x \in A \). If \( f = g + h \) with \( g \in L^p \) and \( h \in L^p \), then \( |g(x)| + |h(x)| \geq t \) for a.e. \( x \in A \), and thus either \( |x \in \mathbb{R}^n : |g(x)| \geq t/2 | \geq |A|/2 \), or \( |x \in \mathbb{R}^n : |h(x)| \geq t/2 | \geq |A|/2 \), or both.

With the convention that \( 1/p = 0 \) if \( p = \infty \), in either case, we have

\[
\|g\|_{L^p} + \|h\|_{L^p} \geq (t/2) \min\{(|A|/2)^{1/p}, (|A|/2)^{1/p}\}.
\]

Since \( g \) and \( h \) were arbitrarily chosen, we conclude that \( \|f\| \geq (t/2) \min\{(|A|/2)^{1/p}, (|A|/2)^{1/p}\} > 0 \).

- \( \|cf\| = |c|\|f\| \) means that

\[
\inf\{\|g\|_{L^p} + \|h\|_{L^p} : cf = g + h\} = |c| \inf\{\|g\|_{L^p} + \|h\|_{L^p} : f = g + h\}.
\]

If \( c = 0 \), this is easy. If \( c \neq 0 \), then for both directions we do a simple scaling by \( c^{-1} \).

- To show the triangle inequality, let \( f_1, f_2 \) be given and let \( g_1 + h_1 = f_1, g_2 + h_2 = f_2 \) be any decomposition. We aim to show that there is a decomposition \( f_1 + f_2 = g_3 + h_3 \) with

\[
\|g_1\|_{L^p} + \|h_1\|_{L^p} + \|g_2\|_{L^p} + \|h_2\|_{L^p} \geq \|g_3\|_{L^p} + \|h_3\|_{L^p}.
\]

Taking \( g_3 = g_1 + g_2 \) and \( h_3 = h_1 + h_2 \), we are done.

b) It is complete. We will prove the completeness using the following criterion in functional analysis: a normed vector space \( X \) is complete if and only if every absolutely convergent sequence is convergent, that is, for each sequence \( f_n \in X \) with \( \sum_n \|f_n\| < \infty \), there is \( f \in X \) such that \( \sum_n f_n = f \) in norm.

Now let \( \sum_n \|f_n\| < \infty \). By definition of \( \|\cdot\| \), for each \( n \) there is \( g_n, h_n \) with \( f_n = g_n + h_n \) such that

\[
\|f_n\| + 2^{-n} > \|g_n\|_{L^p} + \|h_n\|_{L^p}.
\]

Summing in \( n \) we get

\[
\sum_n \|g_n\|_{L^p} + \|h_n\|_{L^p} < \sum_n \|f_n\| + 1 < \infty.
\]
Similarly, there is a constructive method of finding a counter-example for the bonus question in Q3 when \( n = 2 \). In fact, we will construct an \( f \) supported on \([0,1]^2\) with \( f \in L^p([0,1]^2) \) if and only if \( p < 2 \), which is sharp except at the endpoint. It remains unknown to me if we can find \( f \in L^p_{loc}(\mathbb{R}^2) \) with this property.

### Appendix

We give a counter-example for the bonus question in Q3 when \( n = 2 \). In fact, we will construct an \( f \) supported on \([0,1]^2\) with \( f \in L^p([0,1]^2) \) if and only if \( p < 2 \), which is sharp except at the endpoint. It remains unknown to me if we can find \( f \in L^p_{loc}(\mathbb{R}^2) \) with this property.
Main difficulty: one has to construct a function \( f \) and a set \( E \) of positive Lebesgue measure so that \( f \) is not continuous or bounded in any tiny neighbourhood of any \( x \in E \).

Outline of my idea: we first construct a Cantor-like set \( C \) in \([0,1]\) but with positive Lebesgue measure. Then \( C^c \) is a union of countably many open intervals \( \{I_j\} \) with disjoint closures. Thus \( C^c \times C^c \) is a union of countably many open rectangles \( \{T_{j,k}\} \) with disjoint closures. We then define a function \( f \) which takes some suitable value \( a_{j,k} \) on \( T_{j,k} \) such that \( f \in L^p([0,1]^2) \) for all \( 0 < p < 2 \). Next, using the geometry of the circle, we will show \( \lim \sup_{r \to 0+} A_r f(x) > 0 = f(x) \) for all \( x \in C \times C \). Since \( C \times C \) has positive 2 dimensional Lebesgue measure, we are done.

### 0.1 A General Decimal Expansion

For \( k \geq 1 \), let \( M_k \geq 3 \) be integers. Then each real number \( x \in [0,1] \) has the following general decimal expansion corresponding to \( \{M_k\} \):

\[
x = \sum_{k=1}^{\infty} \frac{x_k}{\prod_{j=1}^{k} M_j}, \quad x_k \in \{0,1,\ldots,M_k-1\}.
\]

Each \( x \) has at most 2 different expansions. Moreover, the numbers \( x \in [0,1) \) with 2 different expansions are exactly those which has a terminating decimal expansion; if a number cannot be represented as a terminating decimal number, then it has a unique general decimal expansion.

For an example, let us take \( M_k = 10 \) for all \( k \), which corresponds to our usual decimal expansion. The number 0.1 can also be represented as 0.099999 \ldots , so 0.1 has two decimal expansions. In comparison, the number \( \pi/10 \) has only one decimal expansion.

We are interested in the subset \( C_k \subseteq [0,1] \) that consists of all numbers \( x \in [0,1] \) such that either decimal expansion of \( x \) has \( x_k \neq 1 \). For example, if we take \( M_k = 3 \) for all \( k \), then \( C_1 \) is exactly the union of two closed intervals \([0,1/3] \cup [2/3,1]\). Specially, notice that the number 1/3 can be represented by

\[
\frac{1}{3} = 0.1(3) = 0.02222222\ldots(3),
\]

and since the latter expansion does not have a 1 on the first decimal place, we include 1/3 \( \in C_1 \).

Having constructed \( C_k \), we then consider the general Cantor set \( C \) defined by \( C = \cap_{k=1}^{\infty} C_k \). Equivalently, \( C \) consists of the numbers \( x \in [0,1] \) such that either decimal expansion of \( x \) omits the digit 1 entirely. For instance, if \( M_k = 3 \) for all \( k \), then this \( C \) is exactly the standard middle-third Cantor set on the real line. In the general setting, \( C \) will still be an uncountable, compact, perfect and totally disconnected set. Lastly, note that \( \{0,1\} \subseteq C \) and the complement of \( C \) in \([0,1]\) is a countable union of open intervals with disjoint closures.

The key difference from the standard case is that now \( C \) may have positive Lebesgue measure. More precisely, we have the following criterion:

**Proposition 1.** Let \( M_k \geq 3 \) and \( C \) be the set of the numbers \( x \in [0,1] \) such that either decimal expansion of \( x \) corresponding to \( \{M_k\} \) omits the digit 1 entirely. Then \( |C| > 0 \) if
and only if
\[ \sum_{k=1}^{\infty} \frac{1}{M_k} < \infty. \]

**Proof.** We have
\[ |C_1| = 1 - \frac{1}{M_1}, \quad |C_1 \cap C_2| = \left(1 - \frac{1}{M_1}\right) \left(1 - \frac{1}{M_2}\right), \]
and in general,
\[ |\cap_{j=1}^k C_j| = \prod_{j=1}^k \left(1 - \frac{1}{M_j}\right). \]

Hence \(|C| > 0\) if and only if \(\sum_{k=1}^{\infty} \frac{1}{M_k} < \infty\) by the following elementary lemma in analysis.

\[ \square \]

**Lemma 1.** Let \(\epsilon_k \in (0, 1)\) be a sequence. Then \(\prod_{k=1}^{\infty} (1 - \epsilon_k) > 0\) if and only if \(\sum_{k=1}^{\infty} \epsilon_k < \infty\).

The proof of the lemma is an easy exercise using \(\log(1 - x) \sim x\) for \(x \in (0, 1/2)\).

### 0.2 Construction of the Function

We start with some notation. With \(M_k\) to be specified below, we recall the construction of \(C\) depending on \(M_k\).

Let \(S_k = \prod_{j=1}^{k} M_j\) \((S_0 := 0)\). At the \(k\)-th level of construction, we are removing an open interval from each of the \(S_{k-1}\) remaining closed intervals that make up \(\cap_{j=1}^{k-1} C_j\). We will call the open intervals being removed at the \(k\)-th stage \(I_{k,l}\), where \(1 \leq l \leq S_{k-1}\). Let \(I\) denote the countable collection of all such intervals \(I_{k,l}\), \(k \geq 1\), \(1 \leq l \leq S_{k-1}\). At the same level \(k\), each \(I_{k,l}\) has the same length \(l_k = S_{k-1}^{-1}\).

We need to upgrade the previous setting to \(\mathbb{R}^2\). For each pair of intervals \((I \times J) \in I \times I\), we consider the open rectangle \(T_{I,J} = I \times J\). We observe that \(\{T_{I,J} : (I, J) \in I \times I\}\) is a countable collection of open rectangles with disjoint closures.

We now specify the parameters we will use.

Define
\[ M_k = 2^{k+1}. \tag{2} \]

Then \(\sum_k M_k^{-1} < \infty\) and thus \(|C| > 0\).

Define
\[ f(x) = \sum_{I,J \in I} a_{I,J} 1_{T_{I,J}} \]
where
\[ a_{I,J} = \begin{cases} M_i, & \text{if } I = I_{i,l}, J = I_{i,l'}, \text{ for some } i, l, l', \\ 0, & \text{otherwise}. \end{cases} \]

Thus only perfect squares \(T_{I,J}\) will contribute to the mass of \(f\).
Note that at level $k$ of construction, we remove fewer than $S_{k-1}$ intervals $I_{k,l}$ having the same length $S_k^{-1}$.

Thus

$$\|f\|_p^p \leq \sum_{k=1}^{\infty} S_{k-1}^2 M_k^p S_k^{-2} = \sum_{k=1}^{\infty} M_k^{p-2},$$

which is finite if and only if $p < 2$.

### 0.3 Proof of Divergence

Let $(x, y) \in C \times C$. Then by definition, $f(x, y) = 0$. We will show that $\limsup_{r \to 0^+} A_r f(x, y) > 0$ by choosing a suitable subsequence $r_k$. Since $C \times C$ has positive Lebesgue measure, this proves our claim.

#### 0.3.1 A simple case

As a simple example, let us assume $x = y = 0$ to illustrate the main idea.

Fix $k \geq 1$. We consider the interval $I = I_{k,1} = (S_k^{-1}, 2S_k^{-1}) \in \mathcal{I}$, that is, the collection of all $x \in [0, 1]$ of the form (except for the endpoints of $I$)

$$x = 0.0000 \cdots 01\cdots, \text{(the } k\text{-th digit is 1)}$$

Let $J = I$ as well. Consider the circle $\gamma$ centred at 0 that passes through the upper-left corner of the square $I \times J$. Then the length of the arc inside $I \times J$ is comparable to $S_k^{-1}$.

Thus we have $r_k = \sqrt{S_k^{-2} + (2S_k^{-1})^2} = \sqrt{3}S_k^{-1}$. This implies

$$A_{r_k} f(0, 0) = (2\pi r_k)^{-1} \int_{\gamma} f(x, y) dx dy \geq r_k^{-1} S_k^{-1} a_{I,J} \sim M_k > 1.$$

#### 0.3.2 The general case

Now we come to the general case where $(x, y) \in C \times C$. Write

$$x = \sum_{k=1}^{\infty} \frac{x_k}{S_k}, \quad y = \sum_{k=1}^{\infty} \frac{y_k}{S_k}, \quad x_k, y_k \in \{0, 1, \ldots, M_k - 1\},$$

since $a_{I,J} = M_k$.

Fix $k \geq 1$. We take $I = (x - (x_k - 1)S_k^{-1}, x - (x_k - 2)S_k^{-1}) \in \mathcal{I}$, and $J = (y - (y_k - 1)S_k^{-1}, y - (y_k - 2)S_k^{-1}) \in \mathcal{I}$. Let $\gamma$ be the circle centred at $(x, y)$ that passes through a vertex of $I \times J$ that is neither the closest nor the farthest to $(x, y)$; this ensures that the
arc inside the square $I \times J$ has length comparable to $S_k^{-1}$. We have 4 cases according as whether $x_k = 0$ and whether $y_k = 0$, but in all the cases we have

$$r_k \leq \sqrt{\max\{(x_k - 1)^2, (x_k - 2)^2\} + \max\{(y_k - 1)^2, (y_k - 2)^2\}}S_k^{-1}
\lesssim M_k S_k^{-1} = S_k^{-1}.$$

Let $\gamma = C_{r_k}(x, y)$ denote the circle with centre $(x, y)$ and radius $r_k$.

Hence we have

$$A_{r_k}f(x, y) = (2\pi r_k)^{-1} \int_{\gamma} f(u, v)dudv
\gtrsim r_k^{-1} S_k^{-1} a_{I, J}
\sim M_k^{-1} M_k = 1,$$

since $a_{I, J} = M_k$. 