Math 404/541 Homework 1

- Due Friday Oct 18 at 12:00pm.
- Homework should be submitted using Canvas.
- **Collaboration Policy**: You are welcome (and encouraged) to work on the homework in groups. However, each student must write up the homework on their own, and must use their own wording (i.e. don't jusy copy the solutions from your friend). If you do collaborate with others, please list the name of your collaborators at the top of the homework.
- Homework should be typeset in LaTeX. If you are unfamiliar with LaTeX, "The Not So Short Introduction to LaTeX" is a good place to start. This guide can be found at http://tug.ctan.org/info/lshort/english/lshort.pdf . You can also download the .tex source file for this homework and take a look at that.
- Each homework problem should be correct as stated. Occasionally, however, I might screw something up and give you an impossible homework problem. If you believe a problem is incorrect, please email me. If you are right, the first person to point out an error will get +1 on that homework, and I will post an updated version.

In the problems that follow, Mf refers to the Hardy Littlewood maximal function.

1. a) Prove that $Mf \in L^1(\mathbb{R}^n)$ if and only if $f = 0$ a.e.

b) If $f \in L^1(\mathbb{R}^n)$, it need not even be true that $Mf \in L^1_{\text{loc}}(\mathbb{R}^n)$. Find an example of a function $f \in L^1(\mathbb{R}^n)$ so that $Mf \notin L^1_{\text{loc}}(\mathbb{R}^n)$, and show that your example is correct. Hint: the statement of part (c) might be useful for your search.

c) Let $f\colon \mathbb{R}^n \to \mathbb{C}$ and suppose that

$$\int |f(x)| \log(2 + |f(x)|) dx < \infty.$$

Prove that $Mf \in L^1_{\text{loc}}(\mathbb{R}^n)$.

Hint: If K is a compact set, then

$$\int_K Mf(x)dx = \int_{K\cap\{|Mf|\leq 1\}} Mf(x)dx + \int_{Mf\geq 1} Mf(x)dx \leq |K| + \int_{Mf\geq 1} Mf(x)dx.$$

2. Let $\phi\colon \mathbb{R}^n \to [0,\infty]$ be radial and non-increasing, i.e. $\phi(x) = \phi(y)$ whenever $|x| = |y|$, and $\phi(x) \le \phi(y)$ whenever $|x| \ge |y|$.

Prove that if $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, then $|\phi * f(x)| \le \|\phi\|_1 \, Mf(x)$ for all $x \in \mathbb{R}^n$ with $Mf(x) < \infty$.

3. If $f \in C_c(\mathbb{R}^n)$ and $r > 0$, define

$$A_r f(x) = \int_{S^{n-1}} f(x + ry) d\sigma(y),$$

where S^{n-1} is the $d-1$ dimensional unit sphere in $\mathbb{R}^n$, and σ is normalized surface measure on S^{n-1} (i.e. $\sigma(S^{n-1}) = 1$). Define

$$M_S f(x) = \sup_{r>0} A_r|f|(x).$$

M_S is called the spherical maximal operator. Stein proved that if $n \ge 3$ and $n/(n-1) < p \le \infty$, then there is a constant $C_{n,p}$ so that

$$\|M_S f\|_p \le C_{n,p} \|f\|_p \quad \text{for all } f \in L^p(\mathbb{R}^n) \cap C_c(\mathbb{R}^n). \tag{1}$$

Later, Bourgain proved that (1) holds when $n = 2$ and $2 < p < \infty$.

Using this result, prove that if $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ for the same range of p as above, then

$$\lim_{r \searrow 0} A_r f(x) = f(x) \quad \text{a.e.}$$

Bonus. Previously, this problem asked you to prove that

$$\lim_{r \searrow 0} A_r f(x) = f(x) \quad \text{a.e.}$$

for all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Give a counter-example showing that this is not true.

4. In lecture we discussed the vector space $L^{p_1}(\mathbb{R}^n) + L^{p_2}(\mathbb{R}^n)$ consisting of all equivalence classes of measurable functions that can be written as $g + h$, where $g \in L^{p_1}(\mathbb{R}^n)$ and $h \in L^{p_2}(\mathbb{R}^n)$. If $f \in L^{p_1}(\mathbb{R}^n) + L^{p_2}(\mathbb{R}^n)$, define

$$\|f\|_{L^{p_1}+L^{p_2}} = \inf\{\|g\|_{L^{p_1}} + \|h\|_{L^{p_2}} : f = g + h\}.$$

a) Prove that with this definition, $L^{p_1} + L^{p_2}$ is a normed vector space, i.e. prove that $\|\cdot\|_{L^{p_1}+L^{p_2}}$ is indeed a norm.

b) Is this normed vector space complete? Prove that your answer is correct.

5. For $x \in \mathbb{R}^n$ and $t > 0$, let

$$P_t(x) = \frac{c_n t}{(t^2 + |x|^2)^{\frac{n+1}{2}}}, \quad c_n = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}}.$$

The constant c_n was chosen so that $\int P_t(x) = 1$ for each $t > 0$ (you can take this fact as given; you don't need to prove it). $P_t(x)$ is known as the Poisson kernel.

a) Let $f \in L^p(\mathbb{R}^n)$ with $1 \le p < \infty$. Prove that $\lim_{t\to 0} \|P_t * f - f\|_{L^p(\mathbb{R}^n)} = 0$.

b) Let $f \in L^p(\mathbb{R}^n)$. For each each $(x,t) \in \mathbb{R}^{n+1}_+ = \{(x,t) \in \mathbb{R}^{n+1} \colon t > 0\}$ (this set is called the upper half-space), define $u(x,t) = P_t * f(x)$. Prove that u is harmonic on the upper half-space, i.e. $\Delta u(x,t) = 0$ for all $(x,t) \in \mathbb{R}^{n+1}_+$.

Remark. If $f \colon \mathbb{R}^n \to \mathbb{C}$, then the problem of of finding a harmonic function u on the upper half-space that satisfies $\lim_{(x',t)\to x} u(x',t) = f(x)$ (or some suitable variant of this statement) is known as Dirichlet's problem for the upper half-space.