1. We will first prove the coefficient of $k^n$ is 1 AND the coefficient of $k^{n+r}$ for $r > 0$ is 0. This will be a strong induction on the number of edges $E$.

**Base case:** $E = 0$. $P_G(k) = k^n$ from the notes.

**Induction step:** Assume the result is true for up to and including $m - 1$ edges. Then for a graph $G$ with $m$ edges, by deletion-contraction on some $e \in E(G)$ we have

$$P_G(k) = P_{G-e}(k) - P_{G/e}(k).$$

By induction the coefficient of $k^n$ is 1 in $P_{G-e}(k)$ and 0 in $P_{G/e}(k)$ and the coefficient of $k^{n+r}$ for $r > 0$ is 0, and the result follows.

We now prove the coefficient of $k^{n-1}$ is $-|E(G)|$ by strong induction on the number of edges $E$.

**Base case:** $E = 0$. $P_G(k) = k^n$ from the notes and the coefficient of $k^{n-1}$ is 0 = $-|E(G)|$.

**Induction step:** Assume the result is true for up to and including $m - 1$ edges. Then for a graph $G$ with $m$ edges, by deletion-contraction on some edge $e \in E(G)$ we have

$$P_G(k) = P_{G-e}(k) - P_{G/e}(k).$$

By induction the coefficient of $k^{n-1}$ in $P_{G-e}(k)$ is $-(m-1)$ and in $P_{G/e}(k)$ is 1 by the first part of this question. Hence the coefficient of $k^{n-1}$ in $P_G(k)$ is $-(m-1) - 1 = -m = -|E(G)|$ as desired.

2. We will do a weak induction on the number of components $C$.

**Base case:** $C = 1$. $P_G(k)$ is the chromatic polynomial of the one component.

**Induction step:** Assume the result is true for a graph with $m$ components. Then for $m+1$ components, in components $C_1, \ldots, C_m$ we know that since no vertex in them is connected to any vertex in $C_{m+1}$, then $C_{m+1}$ can be coloured independently from the other components in $P_{C_{m+1}}(k)$ ways. Hence

$$P_G(k) = P_{C_1,\ldots,C_m}(k)P_{C_{m+1}}(k) = P_{C_1}(k) \cdots P_{C_m}(k)P_{C_{m+1}}(k)$$

by the induction hypothesis, and the result follows.
3. First, if $G = K_n$ then $G$ is $n$ colourable since we can assign a different colour to each vertex. On the other hand, by the pigeonhole principle we must have that any colouring of $G$ using fewer than $n$ colours must assign the same colour to two vertices, which is forbidden since every pair of vertices are adjacent.

Conversely, if $G \neq K_n$ then it contains a vertex $v$ such that $\deg(v) < n - 1$. Consider $G' = G - v$. Since $G'$ has $n - 1$ vertices we can colour it with $n - 1$ colours. Reinsert $v$ into $G'$ to recover $G$. Then since $\deg(v) < n - 1$ there must be at least one of the $n - 1$ colours, $c$, not used to colour a vertex adjacent to $v$. Colour $v$ with colour $c$. Hence $\chi(G) \leq n - 1$, so $\chi(G) \neq n$.

4. We will do a strong induction on the number of edges $E$.

Base case: Let $E = 0$. Then $G = N_n$ for $n \geq 1$ and we proved in lecture that $P_G(k) = k^n$, whose one term alternates in sign.

Induction step: Now assume that the result is true for $0 < E < m$. Now given a simple graph with $E = m$ edges and $n$ vertices, by deletion-contraction we get

$$P_G(k) = P_{G-e}(k) - P_{G/e}(k)$$

where by induction giving us that the coefficients alternate in sign, and knowing that the coefficient of the leading term of the chromatic polynomial is 1 by Question 1 we also know that

$$P_{G-e}(k) = k^n - a_{n-1}k^{n-1} + a_{n-2}k^{n-2} - \cdots + (-1)^n a_0$$

$$P_{G/e}(k) = k^{n-1} - b_{n-2}k^{n-2} + b_{n-3}k^{n-3} - \cdots + (-1)^{n-1} b_0$$

for $a_i, b_i \geq 0$ for all $i$. Hence

$$P_G(k) = k^n - (a_{n-1} + 1)k^{n-1} + (a_{n-2} + b_{n-2})k^{n-2} - \cdots + (-1)^n (a_0 + b_0)$$

and the result follows by induction.

5. At the point where we delete and reinsert a vertex of degree 5 in the five colour theorem we rely on the fact that $K_5$ is not planar. However, when we adjust the proof to delete and reinsert a vertex of degree 4 in proving the four colour theorem we would need that $K_4$ is not planar, but this is not true.

6. Let $\chi(G) = k$ and the deletion of any vertex $v$ yields a graph with a smaller chromatic number, i.e. $\chi(G - v) = m \leq k - 1$. If there exists a vertex $\tilde{v} \in V(G)$ of degree less than $k - 1$ delete it. Then colour the remaining graph in $k - 1$ colours. Reinsert $\tilde{v}$. As it is adjacent to at most $k - 2$ vertices there is at least one of the $k - 1$ colours we could colour it, so $\chi(G) \leq k - 1$. Since this is not true, the degree of every vertex must be at least $k - 1$. 