Math 341: Sign of a permutation

In this note we will define the sign of a permutation. This will be useful for problems 4-8 of HW2.

Let $n$ be a positive integer. A permutation $\pi \in S_n$ is called a transposition if it is of the form $\pi = (a_1 a_2)$, i.e. $\pi$ consists of a single cycle, and this cycle contains just two numbers.

Now let $\pi \in S_n$ be an arbitrary permutation. We will prove in lecture that $\pi$ can always be written as a product of transpositions, i.e. $\pi = \pi_1 \pi_2 \cdots \pi_k$, where $\pi_1, \ldots, \pi_k \in S_n$ are transpositions. We define
\[
\text{sgn}(\pi) = (-1)^k,
\]
i.e. $\text{sgn}(\pi) = 1$ if $k$ is even and $\text{sgn}(\pi) = -1$ if $k$ is odd. It’s not immediately obvious that $\text{sgn}(\pi)$ is well defined. Indeed, $\pi$ can be written as a product of transpositions in many different ways, i.e. we could write $\pi = \pi_1 \pi_2 \cdots \pi_k$ or $\pi = \pi'_1 \pi'_2 \cdots \pi'_k'$, where $\pi_1, \ldots, \pi_k$ are transpositions, and $\pi'_1, \ldots, \pi'_k'$ are transpositions. It need not be the case that $k = k'$, but it is the case that $(-1)^k = (-1)^{k'}$, i.e. $k$ is even if and only if $k'$ is even.