Math 341 Homework 6 Solutions

1. a. Give an example of a family of sets $\mathcal{F} = \{A_1, A_2, A_3, A_4\}$ that satisfies Hall's criterion, for which there exists a unique SDR.

Solution. A (boring) solution would be $A_1 = \{1\}$, $A_2 = \{2\}$, $A_3 = \{3\}$, $A_4 = \{4\}$. A slightly less boring solution would be $\{A_1\} = \{1, 3\}$, $A_2 = \{1, 2, 3\}$, $A_3 = \{1, 2, 3, 4\}$, $A_4 = \{1\}$

b. Give an example of a family of sets $\mathcal{F} = \{A_1, A_2, A_3, A_4, A_5\}$ that satisfies Hall's criterion, for which there exists more than one SDR.

Solution. $A_1 = A_2 = A_3 = A_4 = \{1, 2, 3, 4, 5\}$; every 5-tuple $(x_1, x_2, x_3, x_4, x_5)$ of distinct elements with $x_1, \ldots, x_5 \in [5]$ is a SDR. Thus there are $5! = 120$ SDRs for this family of sets.

2. Let $A_1, \ldots, A_n$ be sets, with $n \geq 3$. Suppose that for every set of indices $I \subset [n]$, we have

$$|A(I)| \geq |I| + 2.$$

Let $x_1 \in A_1$ and $x_2 \in A_2$ with $x_1 \neq x_2$. Prove that there exists $x_3, \ldots, x_n$ so that $(x_1, x_2, x_3, \ldots, x_n)$ is a SDR for $A_1, \ldots, A_n$.

Solution. For each $i = 3, \ldots, n$, define $A_i' = A_i \backslash \{x_1, x_2\}$. Then for each set of indices $I \subset \{3, \ldots, n\}$, we have

$$|A'(I)| \geq |A(I) \backslash \{x_1, x_2\}| \geq |A(I)| - 2 \geq |I|.$$

Thus $A_3', \ldots, A_n'$ satisfies Hall's criterion, so there exists an SDR $(x_3, x_4, \ldots, x_n)$ for $A_3', \ldots A_n'$. Since for each $i = 3, \ldots, n$ we have $x_i \neq x_1$ and $x_i \neq x_2$, we conclude that $(x_1, x_2, x_3, \ldots, x_n)$ is a SDR for $A_1, \ldots, A_n$.

3. Let $n = 7 = 2^2 + 2 + 1$. Write down 7 sets $A_1, \ldots, A_n$ so that $\mathcal{F} = \{A_1, \ldots, A_7\}$ is a family of subsets of $[7]$; each set A_i has cardinality 3; each number $x \in [7]$ is contained in 3 sets, and each pair of sets intersect in exactly one element.

Solution.

The 7 sets are $\{2, 4, 6\}$, $\{1, 4, 5\}$, $\{3, 4, 7\}$, $\{2, 1, 3\}$, $\{2, 5, 7\}$, $\{1, 6, 7\}$, $\{3, 5, 6\}$

4. Define $\mathbb{Z}_2 = \{0, 1\}$; if $a, b \in \mathbb{Z}_2$, we define $a + b = 0$ if $a = 0, b = 0$ or $a = 1, b = 1$. We define $a + b = 1$ if $a = 0, b = 1$ or $a = 1, b = 0$ (this is called addition mod 2, or XOR). If $a, b \in \mathbb{Z}_2$, define $ab = 0$ if $a = 0$ or $b = 0$ (or both), and define $ab = 1$ if $a = 1, b = 1$. With these definitions, $\mathbb{Z}_2$ is called a *ring*.

Let $\mathcal{P} = \mathbb{Z}_2^3 \backslash \{(0, 0, 0)\}$, i.e.

$$\mathcal{P} = \{(0, 0, 1), (0, 1, 0), (0, 1, 1), (1, 0, 0), (1, 0, 1), (1, 1, 0), (1, 1, 1)\}.$$

For each $(a, b, c) \in \mathbb{Z}_2 \backslash \{(0, 0, 0)\}$, define

$$L_{(a,b,c)} = \{(x, y, z) \in \mathcal{P} \colon ax + by + cz = 0\},$$

where addition and multiplication is performed according to the rules described above. For example, if $(a,b,c)=(1,0,1)$, then

$$L_{(1,0,1)}=\{(x,y,z)\in\mathcal{P}\colon x+z=0\}=\{(1,0,1),\ (1,1,1),\ (0,1,0)\}.$$

Let $\mathcal{F}$ be the family of 7 sets

$$\mathcal{F}=\{L_{(a,b,c)}\colon (a,b,c)\in\mathbb{Z}_2\backslash\{0,0,0\}\}.$$

Write down the 7 sets in $\mathcal{F}$ (we wrote down $L_{(1,0,1)}$ above; you need to write down the rest).

Remark: Observe that each pair of sets intersect in exactly one element; each set contains 3 elements, and each element of $\mathcal{P}$ is contained in exactly 3 of the sets from $\mathcal{F}$.

Solution. We have that

$$\mathcal{F}=\{L_{(0,0,1)},L_{(0,1,0)},L_{(0,1,1)},L_{(1,0,0)},L_{(1,0,1)},L_{(1,1,0)},L_{(1,1,1)}\},$$

where

$$L_{(0,0,1)}=\{(0,1,0),(1,0,0),(1,1,0)\},\tag{1}$$
$$L_{(0,1,0)}=\{(0,0,1),(1,0,0),(1,0,1)\},\tag{2}$$
$$L_{(0,1,1)}=\{(0,1,1),(1,0,0),(1,1,1)\},\tag{3}$$
$$L_{(1,0,0)}=\{(0,1,0),(0,0,1),(0,1,1)\},\tag{4}$$
$$L_{(1,0,1)}=\{(0,1,0),(1,0,1),(1,1,1)\},\tag{5}$$
$$L_{(1,1,0)}=\{(0,0,1),(1,1,0),(1,1,1)\},\tag{6}$$
$$L_{(1,1,1)}=\{(0,1,1),(1,0,1),(1,1,0)\}.\tag{7}$$

Observe that these are the "same" sets as in problem 3, if we interpret the element (x,y,z) as the binary number $2^2a+2^1b+2^0c$, i.e. $L_{(0,0,1)}=\{(0,1,0),(1,0,0),(1,1,0)\}$ corresponds to the set $\{2,4,6\}$, which is the first of the 7 sets from problem 3.

5. Let $\mathcal{F}$ be a family of subsets of $[n]$. Suppose that each set in $\mathcal{F}$ has cardinality k, and that for every collection of $k+1$ sets $A_1,\dots,A_{k+1}\in\mathcal{F}$, we have that their intersection $A_1\cap A_2\cap\ldots\cap A_{k+1}$ is non-empty. Prove that the intersection of all the sets in $\mathcal{F}$ is non-empty, i.e. all of the sets in $\mathcal{F}$ contain a common element.

Solution. We will do a proof by contradiction. Let $A=\{x_1,\dots,x_k\}$ be a set from $\mathcal{F}$. If the intersection of all the sets in $\mathcal{F}$ is empty, then there exists a set $A_1\in\mathcal{F}$ with $x_1\notin A_1$. Similarly, for each $i=2,\dots,k$, there exists a set $A_i\in\mathcal{F}$ with $x_i\notin A_i$. But then $A\cap A_1\cap A_2\cap\ldots\cap A_k=\emptyset$, which contradicts the assumption that every collection of $k+1$ sets from $\mathcal{F}$ has non-empty intersection.

6. An intersecting family $\mathcal{F}$ of subsets of $[n]$ is called *maximal* if every larger family $\mathcal{F}'\supsetneq\mathcal{F}$ is *not* intersecting. I.e. it is impossible to add an additional set to $\mathcal{F}$ so that the resulting family is still intersecting.

Prove that every maximal intersecting family of subsets of $[n]$ has cardinality 2^{n-1}.

Solution. Let $\mathcal{F}$ be an intersecting family, and suppose that $|\mathcal{F}|<2^{n-1}$. We will show that there exists a set $B\subset[n]$ that intersects every set in $\mathcal{F}$. This means that $\mathcal{F}\cup\{B\}$ is intersecting, and thus $\mathcal{F}$ is not maximal.

Group the 2^n subsets of $[n]$ into 2^{n-1} complimentary pairs of the form $(A,[n]\backslash A)$. Since $|\mathcal{F}|<2^{n-1}$, there must exist a set A so that neither A nor $[n]\backslash A$ is contained in $\mathcal{F}$. If A intersects every set in $\mathcal{F}$, then let $B=A$ and we are done. If not, then there is a set $C\in\mathcal{F}$ with $C\cap A=\emptyset$. But this means that $C\subset([n]\backslash A)$. Let $B=[n]\backslash A$. Since $C\subset B$ and C intersects every set in $\mathcal{F}$, we have that B also intersects every set in $\mathcal{F}$, and we're done.