Math 341 Homework 6 Solutions

1. a. Give an example of a family of sets \( \mathcal{F} = \{A_1, A_2, A_3, A_4\} \) that satisfies Hall’s criterion, for which there exists a unique SDR.

   Solution. A (boring) solution would be \( A_1 = \{1\}, A_2 = \{2\}, A_3 = \{3\}, A_4 = \{4\} \). A slightly less boring solution would be \( \{A_1\} = \{1, 3\}, A_2 = \{1, 2, 3\}, A_3 = \{1, 2, 3, 4\}, A_4 = \{1\} \).

b. Give an example of a family of sets \( \mathcal{F} = \{A_1, A_2, A_3, A_4, A_5\} \) that satisfies Hall’s criterion, for which there exists more than one SDR.

   Solution. \( A_1 = A_2 = A_3 = A_4 = \{1, 2, 3, 4, 5\} \); every 5-tuple \((x_1, x_2, x_3, x_4, x_5)\) of distinct elements with \(x_1, \ldots, x_5 \in [5]\) is a SDR. Thus there are \(5! = 120\) SDRs for this family of sets.

2. Let \( A_1, \ldots, A_n \) be sets, with \( n \geq 3 \). Suppose that for every set of indices \( I \subset [n] \), we have

   \[ |A(I)| \geq |I| + 2. \]

   Let \( x_1 \in A_1 \) and \( x_2 \in A_2 \) with \( x_1 \neq x_2 \). Prove that there exists \( x_3, \ldots, x_n \) so that \((x_1, x_2, x_3, \ldots, x_n)\) is a SDR for \( A_1, \ldots, A_n \).

   Solution. For each \( i = 3, \ldots, n \), define \( A'_i = A_i \setminus \{x_1, x_2\} \). Then for each set of indices \( I \subset \{3, \ldots, n\} \), we have

   \[ |A'(I)| \geq |A(I) \setminus \{x_1, x_2\}| \geq |A(I)| - 2 \geq |I|. \]

   Thus \( A'_3, \ldots, A'_n \) satisfies Hall’s criterion, so there exists an SDR \((x_3, x_4, \ldots, x_n)\) for \( A'_3, \ldots, A'_n \). Since for each \( i = 3, \ldots, n \) we have \( x_i \neq x_1 \) and \( x_i \neq x_2 \), we conclude that \((x_1, x_2, x_3, \ldots, x_n)\) is a SDR for \( A_1, \ldots, A_n \).

3. Let \( n = 7 = 2^2 + 2 + 1 \). Write down 7 sets \( A_1, \ldots, A_n \) so that \( \mathcal{F} = \{A_1, \ldots, A_7\} \) is a family of subsets of \([7]\); each set \( A_i \) has cardinality 3; each number \( x \in [7] \) is contained in 3 sets, and each pair of sets intersect in exactly one element.

   Solution.

   The 7 sets are \( \{2, 4, 6\}, \{1, 4, 5\}, \{3, 4, 7\}, \{2, 1, 3\}, \{2, 5, 7\}, \{1, 6, 7\}, \{3, 5, 6\} \).

4. Define \( \mathbb{Z}_2 = \{0, 1\} \); if \( a, b \in \mathbb{Z}_2 \), we define \( a + b = 0 \) if \( a = 0, b = 0 \) or \( a = 1, b = 1 \). We define \( a + b = 1 \) if \( a = 0, b = 1 \) or \( a = 1, b = 0 \) (this is called addition mod 2, or XOR). If \( a, b \in \mathbb{Z}_2 \), define \( ab = 0 \) if \( a = 0 \) or \( b = 0 \) (or both), and define \( ab = 1 \) if \( a = 1, b = 1 \). With these definitions, \( \mathbb{Z}_2 \) is called a ring.

   Let \( \mathcal{P} = \mathbb{Z}_2^3 \setminus \{(0, 0, 0)\} \), i.e.

   \[ \mathcal{P} = \{(0, 0, 1), (0, 1, 0), (0, 1, 1), (1, 0, 0), (1, 0, 1), (1, 1, 0), (1, 1, 1)\}. \]

   For each \((a, b, c) \in \mathbb{Z}_2 \setminus \{(0, 0, 0)\}\), define

   \[ L_{(a,b,c)} = \{(x, y, z) \in \mathcal{P}: ax + by + cz = 0\}, \]

   1
where addition and multiplication is performed according to the rules described above. For example, if \((a, b, c) = (1, 0, 1)\), then

\[
L_{(1,0,1)} = \{(x, y, z) \in \mathcal{P} : x + z = 0\} = \{(1, 0, 1), (1, 1, 1), (0, 1, 0)\}.
\]

Let \(\mathcal{F}\) be the family of 7 sets

\[
\mathcal{F} = \{L_{(a,b,c)} : (a, b, c) \in \mathbb{Z}_2 \setminus \{0, 0\}\}.
\]

Write down the 7 sets in \(\mathcal{F}\) (we wrote down \(L_{(1,0,1)}\) above; you need to write down the rest).

Remark: Observe that each pair of sets intersect in exactly one element; each set contains 3 elements, and each element of \(\mathcal{P}\) is contained in exactly 3 of the sets from \(\mathcal{F}\).

**Solution.** We have that

\[
\mathcal{F} = \{L_{(0,0,1)}, L_{(0,1,0)}, L_{(0,1,1)}, L_{(1,0,0)}, L_{(1,0,1)}, L_{(1,1,0)}, L_{(1,1,1)}\},
\]

where

\[
L_{(0,0,1)} = \{(0, 1, 0), (1, 0, 0), (1, 1, 0)\},
\]

\[
L_{(0,1,0)} = \{(0, 0, 1), (1, 0, 0), (1, 0, 1)\},
\]

\[
L_{(0,1,1)} = \{(0, 1, 1), (1, 0, 0), (1, 1, 1)\},
\]

\[
L_{(1,0,0)} = \{(0, 1, 0), (0, 0, 1), (0, 1, 1)\},
\]

\[
L_{(1,0,1)} = \{(0, 1, 0), (1, 0, 1), (1, 1, 1)\},
\]

\[
L_{(1,1,0)} = \{(0, 0, 1), (1, 1, 0), (1, 1, 1)\},
\]

\[
L_{(1,1,1)} = \{(0, 1, 1), (1, 0, 1), (1, 1, 0)\}.
\]

Observe that these are the “same” sets as in problem 3, if we interpret the element \((x, y, z)\) as the binary number \(2^2a + 2^1b + 2^0c\), i.e. \(L_{(0,0,1)} = \{(0, 1, 0), (1, 0, 0), (1, 1, 0)\}\) corresponds to the set \(\{2, 4, 6\}\), which is the first of the 7 sets from problem 3.

5. Let \(\mathcal{F}\) be a family of subsets of \([n]\). Suppose that each set in \(\mathcal{F}\) has cardinality \(k\), and that for every collection of \(k + 1\) sets \(A_1, \ldots, A_{k+1} \in \mathcal{F}\), we have that their intersection \(A_1 \cap A_2 \cap \ldots \cap A_{k+1}\) is non-empty. Prove that the intersection of all the sets in \(\mathcal{F}\) is non-empty, i.e. all of the sets in \(\mathcal{F}\) contain a common element.

**Solution.** We will do a proof by contradiction. Let \(A = \{x_1, \ldots, x_k\}\) be a set from \(\mathcal{F}\). If the intersection of all the sets in \(\mathcal{F}\) is empty, then there exists a set \(A_1 \in \mathcal{F}\) with \(x_1 \notin A_1\). Similarly, for each \(i = 2, \ldots, k\), there exists a set \(A_i \in \mathcal{F}\) with \(x_i \notin A_i\). But then \(A_1 \cap A_2 \cap \ldots \cap A_k = \emptyset\), which contradicts the assumption that every collection of \(k + 1\) sets from \(\mathcal{F}\) has non-empty intersection.

6. An intersecting family \(\mathcal{F}\) of subsets of \([n]\) is called **maximal** if every larger family \(\mathcal{F'} \supseteq \mathcal{F}\) is not intersecting. I.e. it is impossible to add an additional set to \(\mathcal{F}\) so that the resulting family is still intersecting.

Prove that every maximal intersecting family of subsets of \([n]\) has cardinality \(2^{n-1}\).

**Solution.** Let \(\mathcal{F}\) be an intersecting family, and suppose that \(|\mathcal{F}| < 2^{n-1}\). We will show that there exists a set \(B \subset [n]\) that intersects every set in \(\mathcal{F}\). This means that \(\mathcal{F} \cup \{B\}\) is intersecting, and thus \(\mathcal{F}\) is not maximal.

Group the \(2^n\) subsets of \([n]\) into \(2^{n-1}\) complimentary pairs of the form \((A, [n] \setminus A)\). Since \(|\mathcal{F}| < 2^{n-1}\), there must exist a set \(A\) so that neither \(A\) nor \([n] \setminus A\) is contained in \(\mathcal{F}\). If \(A\) intersects every set in \(\mathcal{F}\), then let \(B = A\) and we are done. If not, then there is a set \(C \in \mathcal{F}\) with \(C \cap A = \emptyset\). But this means that \(C \subset ([n] \setminus A)\). Let \(B = [n] \setminus A\). Since \(C \subset B\) and \(C\) intersects every set in \(\mathcal{F}\), we have that \(B\) also intersects every set in \(\mathcal{F}\), and we’re done.