Math 341 Homework 6

- This problem set will not be graded; it is for practice only.

1. a. Give an example of a family of sets $\mathcal{F} = \{A_1, A_2, A_3, A_4\}$ that satisfies Hall's criterion, for which there exists a unique SDR.

b. Give an example of a family of sets $\mathcal{F} = \{A_1, A_2, A_3, A_4, A_5\}$ that satisfies Hall's criterion, for which there exists more than one SDR.

2. Let $A_1, \ldots, A_n$ be sets, with $n \geq 3$. Suppose that for every set of indices $I \subset [n]$, we have

$$|A(I)| \geq |I| + 2.$$

Let $x_1 \in A_1$ and $x_2 \in A_2$ with $x_1 \neq x_2$. Prove that there exists $x_3, \ldots, x_n$ so that $(x_1, x_2, x_3, \ldots, x_n)$ is a SDR for $A_1, \ldots, A_n$.

3. Let $n = 7 = 2^2 + 2 + 1$. Write down 7 sets $A_1, \ldots, A_n$ so that $\mathcal{F} = \{A_1, \ldots, A_7\}$ is a family of subsets of $[7]$; each set A_i has cardinality 3; each number $x \in [7]$ is contained in 3 sets, and each pair of sets intersect in exactly one element.

4. Define $\mathbb{Z}_2 = \{0, 1\}$; if $a, b \in \mathbb{Z}_2$, we define $a + b = 0$ if $a = 0, b = 0$ or $a = 1, b = 1$. We define $a + b = 1$ if $a = 0, b = 1$ or $a = 1, b = 0$ (this is called addition mod 2, or XOR). If $a, b \in \mathbb{Z}_2$, define $ab = 0$ if $a = 0$ or $b = 0$ (or both), and define $ab = 1$ if $a = 1, b = 1$. With these definitions, $\mathbb{Z}_2$ is called a *ring*.

Let $\mathcal{P} = \mathbb{Z}_2^3 \backslash \{(0,0,0)\}$, i.e.

$$\mathcal{P} = \{(0,0,1), (0,1,0), (0,1,1), (1,0,0), (1,0,1), (1,1,0), (1,1,1)\}.$$

For each $(a, b, c) \in \mathbb{Z}_2 \backslash \{(0,0,0)\}$, define

$$L_{(a,b,c)} = \{(x, y, z) \in \mathcal{P} \colon ax + by + cz = 0\},$$

where addition and multiplication is performed according to the rules described above. For example, if $(a, b, c) = (1, 0, 1)$, then

$$L_{(1,0,1)} = \{(x, y, z) \in \mathcal{P} \colon x + z = 0\} = \{(1,0,1),\ (1,1,1),\ (0,1,0)\}.$$

Let $\mathcal{F}$ be the family of 7 sets

$$\mathcal{F} = \{L_{(a,b,c)} \colon (a, b, c) \in \mathbb{Z}_2 \backslash \{0,0,0\}\}.$$

Write down the 7 sets in $\mathcal{F}$ (we wrote down $L_{(1,0,1)}$ above; you need to write down the rest).

Remark: Observe that each pair of sets intersect in exactly one element; each set contains 3 elements, and each element of $\mathcal{P}$ is contained in exactly 3 of the sets from $\mathcal{F}$.

5. Let $\mathcal{F}$ be a family of subsets of $[n]$. Suppose that each set in $\mathcal{F}$ has cardinality k, and that for every collection of $k+1$ sets $A_1, \ldots, A_{k+1} \in \mathcal{F}$, we have that their intersection $A_1 \cap A_2 \cap \ldots \cap A_{k+1}$ is non-empty. Prove that the intersection of all the sets in $\mathcal{F}$ is non-empty, i.e. all of the sets in $\mathcal{F}$ contain a common element.

6. An intersecting family $\mathcal{F}$ of subsets of $[n]$ is called *maximal* if every larger family $\mathcal{F}' \supsetneq \mathcal{F}$ is *not* intersecting. I.e. it is impossible to add an additional set to $\mathcal{F}$ so that the resulting family is still intersecting.

Prove that every maximal intersecting family of subsets of $[n]$ has cardinality 2^{n-1}.