Math 341 Homework 5 Solutions

1. Prove that for each positive integer n, there are about $\sqrt{8n/3}$ pentagonal numbers less than n. More precisely, prove that for every $\epsilon > 0$, there is a number N so that for all $n \geq N$, the number of pentagonal numbers less than n is at most $(1+\epsilon)\sqrt{8n/3}$ and is at least $(1-\epsilon)\sqrt{8n/3}$.

Solution. Recall that a number is pentagonal if it can be written in the form $k(3k-1)/2$ or $k(3k+1)/2$, where $k \geq 0$. The pentagonal number 0 can be represented in two ways (as $0(3\cdot 0-1)/2$ or $0(\cdot 0+1)/2$), but every other pentagonal number has a unique representation.

If $n = s(3s-1)/2$ for some positive integer s, then there are exactly $s = (1+\sqrt{1+24n})/6$ numbers of the form $k(3k-1)/2$ that are less than n. If n is an arbitrary positive integer, then there are $\lceil (1+\sqrt{1+24n})/6 \rceil$ numbers of the form $k(3k-1)/2$ that are less than n. Similarly, if n is a positive integer, then there are $\lceil(-1+\sqrt{1+24n})/6\rceil$ numbers of the form $k(3k+1)/2$ that are less than n

Thus, if n is a positive integer, the number of pentagonal numbers less than n is $\lceil (1+\sqrt{1+24n})/6\rceil + \lceil(-1+\sqrt{1+24n})/6\rceil - 1$.

Now, let $\epsilon > 0$, and let $N = \max\left(\frac{1}{12\epsilon}, \frac{3}{2\epsilon^2}\right)$. With this choice of N, we have that if $n \geq N$, then $1 + 24n \leq (1+\epsilon/2)24n$ (since $n \geq \frac{1}{12\epsilon}$), and $1 \leq (\epsilon/2)\sqrt{8n/3}$ (since $n \geq \frac{27}{2\epsilon^2}$).

In the computations below, we will use the estimate that if $x > 0$, then $1 \leq \sqrt{1+x} \leq 1+x$. We have the following estimates:

$$\lceil (1+\sqrt{1+24n})/6\rceil \leq (1+\sqrt{1+24n})/6 + 1 \leq 7/6 + \sqrt{2/3}(1+\epsilon/2)\sqrt{n}.$$

Similarly,

$$\lceil(-1+\sqrt{1+24n})/6\rceil \leq 5/6 + \sqrt{2/3}(1+\epsilon/2)\sqrt{n}.$$

Thus

$$\lceil (1+\sqrt{1+24n})/6\rceil + \lceil(-1+\sqrt{1+24n})/6\rceil - 1 \leq (1+\epsilon/2)\sqrt{8n/3} + 1 \leq (1+\epsilon)\sqrt{8n/3}.$$

On the other hand, we have

$$\lceil (1+\sqrt{1+24n})/6\rceil \geq 1 + \sqrt{1+24n})/6 \geq 1 + \sqrt{2}3\sqrt{n},$$

and

$$\lceil(-1+\sqrt{1+24n})/6\rceil \geq -1 + \sqrt{1+24n})/6 \geq -1\sqrt{2}3\sqrt{n},$$

so

$$\lceil (1+\sqrt{1+24n})/6\rceil + \lceil(-1+\sqrt{1+24n})/6\rceil - 1 \geq \sqrt{8n/3} - 1 \geq (1-\epsilon)\sqrt{8n/3}.$$

2 a. Let $a_1, a_2, ..., a_n$ be a sequence of real numbers, with $a_1 \geq a_2 \geq \ldots \geq a_n \geq 0$. Prove that

$$\sum_{k=1}^{n} (-1)^{k-1} a_k \leq a_1.$$

Solution. If n is odd, write

$$\sum_{k=1}^{n} (-1)^{k-1} a_k = a_1 - (a_2 - a_3) - (a_4 - a_5) - \ldots - (a_{n-1} - a_n).$$

Since $a_{i+1} \leq a_i$ for each index i, every term in brackets is non-negative, so the entire sum is bounded by a_1.

Similarly, if n is even, write

$$\sum_{k=1}^{n}(-1)^{k-1}a_k = a_1 - (a_2 - a_3) - (a_4 - a_5) - \ldots - (a_{n-2} - a_{n-1}) - (a_n).$$

Again, every term in brackets is non-negative, so the entire sum is bounded by a_1.

b. Prove that for $n \geq 5$, $p(n) \leq F_{n+1}$, where $p(n)$ is the partition function of n and F_n is the n–th Fibonacci number (recall that $F_1 = 1, F_2 = 1$, and $F_3 = 2$). Hint: part a and induction might be helpful.

Solution. If $n = 5$, then $p(5) = 7$, while $F_6 = 8$. If $n = 6$ then $p(6) = 11$, while $F_7 = 13$. Now suppose that $n \geq 7$ and that we have established the inequality $p(k) \leq F_{k+1}$ for all $5 \leq k \leq n-1$. For each $\ell = 1, \ldots, n$, define

$$a_\ell = \big(p(n - \frac{1}{2}\ell(3\ell - 1)) + p(n - \frac{1}{2}\ell(3\ell + 1))\big)$$

Next, note that the partition function $p(s)$ is non-decreasing, since there is an injection from the set of partitions of s to the set of partitions of $s+1$: for every partition $\lambda = 1^{a_1}2^{a_2}\cdots s^{a_s}$ of s, we associate the partition $1^{a_1+1}2^{a_2}\cdots s^{a_s}$ of $s+1$. This means that $a_1 \geq a_2 \geq a_3 \geq \ldots$. Thus using part a, we have

$$p(n) = \sum_{\ell=1}^{n}(-1)^{\ell-1}a_\ell \geq a_1 = p(n-1) + p(n-2) \geq F_n + F_{n-1} = F_{n+1}.$$

3. To state this problem, we need to introduce some notation. Let n be a positive integer and let λ be a partition of n. Write $\lambda = 1^{a_{1,\lambda}}2^{a_{2,\lambda}}\cdots n^{a_{n,\lambda}}$.

With this notation, prove that for each positive integer n,

$$\sum_{\lambda \text{ a partition of n}} \Big(\prod_{i=1}^{n} i^{a_{i,\lambda}} a_{i,\lambda}!\Big)^{-1} = 1.$$

Solution. Recall that for each partition $\lambda = 1^{a_{1,\lambda}}2^{a_{2,\lambda}}\cdots n^{a_{n,\lambda}}$ of n, the number of permutations in S_n with cycle structure λ is $\frac{n!}{\prod_{k=1}^{n} k^{a_{k,\lambda}} a_{k,\lambda}!}$. Since each permutation in S_n has a unique cycle structure (which corresponds to some partition of n), we conclude that

$$\sum_{\lambda \text{ a partition of n}} \frac{n!}{\prod_{i=1}^{n} i^{a_{i,\lambda}} a_{i,\lambda}!} = |S_n| = n!$$

Dividing both sides by $n!$ yields the desired inequality.

4. Prove that for $k > n/2$, the number of permutations in S_n that have a cycle of length k is $n!/k$.

Solution. Observe that if $\pi \in S_n$ has a cycle of length k, then it has exactly one cycle of length k, and all other cycles have length less than k (this is where we use the fact that $k > n/2$). Thus we can construct all permutations containing a cycle of length k as follows: Choose k elements $a_1, \ldots, a_k$ of $[n]$; there are $\binom{n}{k}$ ways to do this. We know from the midterm that there are $(k-1)!$ cycles containing $a_1, \ldots, a_k$ (indeed, there are $(k-1)!$ cyclic orderings of $a_1, \ldots, a_k$). Next, there are $(n-k)!$

permutations of the remaining numbers $[n]\backslash\{a_1,\ldots,a_k\}$. Thus there are $\binom{n}{k}(k-1)!(n-k)! = n!/k$ permutations in S_n. that contain a cycle of length k.

5. Three candidates, A, B, and C, are running in an election. Suppose that $3n$ voters secretly cast their votes by writing one candidate's name on a ballot, and then putting it into a ballot box. The ballot only contains the candidate's name; no other information. The result is a three-way tie: each candidate receives exactly n votes.

The votes are counted as follows: an election official takes ballots out of the box one by one, and keeps a running tally of how many votes each candidate has received at that point. We know before any votes are counted, each candidate has received 0 votes, and after all $3n$ votes are counted, each candidate has received n votes.

How many different ways can the ballots be removed from the box so that at every point, the number of votes for candidate A is at least as large as the number of votes for candidate B, and the number of votes for candidate B is at least as large as the number of votes for candidate C? Hint: politicians and clowns aren't so different.

Solution. If the ballots are removed from the box so that at every point, the number of votes for candidate A is at least as large as the number of votes for candidate B, and the number of votes for candidate B is at least as large as the number of votes for candidate C, then we will call this an "admissible" way of counting the ballots.

Consider the Young diagram $D(\lambda)$ corresponding to the partition $\lambda = n^3$ (this is a partition of $3n$). This diagram has three rows, each of which contains n columns. We will show that every Young tableau with diagram $D(\lambda)$ corresponds to an admissible way of counting the ballots.

Suppose that for each $i = 1,\ldots,3n$, the i–th ballot removed from the box is for candidate $f(i)$ (so $f(i) \in \{1,2,3\}$). If this ballot counting is admissible, then we will create a Young tableau as follows: row j contains the numbers $\{i\colon f(i) = j\}$, in increasing order. Clearly we have written the numbers $1,\ldots,3n$ into the $3n$ boxes of the Young diagram $D(\lambda)$, and the numbers are increasing as we move from left to right. The numbers are also increasing as we move from top to bottom, since at every stage of the process, the number of votes candidate A is at least as large as the number of votes for candidate B, and the number of votes for candidate B is at least as large as the number of votes for candidate C. We have just described an injective function from the set of admissible ways of counting ballots to the set of Young tableau. The function is also bijective, since if Y is a Young tableau with diagram $D(\lambda)$, then Y corresponds to the admissible ballot count where $f(i)$ is the row of Y in which the number i appears.

We will use the hook-length formula to count how many Young tableau have diagram $D(\lambda)$. The cells in row 1 have hook length $h(1,j) = 3+n-j$; the cells in row 2 have hook length $h(2,j) = 2+n-j$, and the cells in row 3 how hook length $h(3,j) = 1+n-j$. Thus by the hook-length formula, the number of Young tableau with diagram $D(\lambda)$ is

$$\frac{(3n)!}{\prod h(i,j)} = \frac{(3n)!}{\left(\prod_{j=1}^{n}(3+n-j)\right)\left(\prod_{j=1}^{n}(2+n-j)\right)\left(\prod_{j=1}^{n}(1+n-j)\right)} = \frac{2(3n)!}{(n+2)!(n+1)!n!}.$$

6. How many Young tableau are associated to each of the following partitions?

a. $\lambda = 1^3 2^2 4^1$.

Solution. We will draw the Young diagram $D(\lambda)$ and the hook-length of each cell.

9	5	2	1
6	2		
5	1		
3			
2			
1			

Since $n = 11$, the number of Young tableau with diagram $D(\lambda)$ is

$$\frac{11!}{9 \cdot 5 \cdot 2 \cdot 1 \cdot 6 \cdot 2 \cdot 5 \cdot 1 \cdot 3 \cdot 2 \cdot 1} = 1232.$$

b. $\lambda = 3^2 4^2 5^1$.

Solution. We will draw the Young diagram $D(\lambda)$ and the hook-length of each cell.

9	8	7	4	1
7	6	5	2	
6	5	4	1	
4	3	2		
3	2	1		

Since $n = 19$, the number of Young tableau with diagram $D(\lambda)$ is

$$\frac{19!}{9 \cdot 8 \cdot 7 \cdot 4 \cdot 1 \cdot 7 \cdot 6 \cdot 5 \cdot 2 \cdot 6 \cdot 5 \cdot 4 \cdot 1 \cdot 4 \cdot 3 \cdot 2 \cdot 3 \cdot 2 \cdot 1} = 8\ 314\ 020.$$

c. $\lambda = 2^1 3^3 4^1$.

Solution. We will draw the Young diagram $D(\lambda)$ and the hook-length of each cell.

8	7	5	1
6	5	3	
5	4	2	
4	3	1	
2	1		

Since $n = 15$, the number of Young tableau with diagram $D(\lambda)$ is

$$\frac{15!}{8 \cdot 7 \cdot 5 \cdot 1 \cdot 6 \cdot 5 \cdot 3 \cdot 5 \cdot 4 \cdot 2 \cdot 4 \cdot 3 \cdot 1 \cdot 2 \cdot 1} = 54054.$$

7. For each of $n = 1, 2, 3, 4$, write down all of the partitions of n, all of the Young Tableau associated to each partition, and compute the quantity

$$\sum_{\lambda} f_\lambda^2,$$

where the sum is taken over all partitions λ of n (recall that f_λ is an integer; f_λ^2 means the square of this integer).

Remark: there appears to be a pattern here. Do you have a conjecture for what happens for larger values of n?

Solution. $\mathbf{n = 1}$:

Partition	Young Tableau	f_λ
1^1	$\boxed{1}$	1

$$\sum_{\lambda} f_\lambda^2 = 1^1 = 1.$$

n = 2:

Partition	Young Tableau	f_λ
1^2	$\begin{array}{\|c\|}\hline 1\\\hline 2\\\hline\end{array}$	1
2^1	$\begin{array}{\|c\|c\|}\hline 1&2\\\hline\end{array}$	1

$$\sum_{\lambda} f_\lambda^2 = 1^1 + 1^2 = 2.$$

n = 3:

Partition	Young Tableau	f_λ
1^3	$\begin{array}{\|c\|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array}$	1
$1^1 2^1$	$\begin{array}{\|c\|c\|}\hline 1&2\\\hline 3\\\cline{1-1}\end{array}$ $\begin{array}{\|c\|c\|}\hline 1&3\\\hline 2\\\cline{1-1}\end{array}$	2
3^1	$\begin{array}{\|c\|c\|c\|}\hline 1&2&3\\\hline\end{array}$	1

$$\sum_{\lambda} f_\lambda^2 = 1^1 + 2^2 + 1^2 = 6.$$

n = 4:

Partition	Young Tableau	f_λ
1^4	$\begin{array}{\|c\|}\hline 1\\\hline 2\\\hline 3\\\hline 4\\\hline\end{array}$	1
$1^2 2^1$	$\begin{array}{\|c\|c\|}\hline 1&2\\\hline 3\\\cline{1-1} 4\\\cline{1-1}\end{array}$ $\begin{array}{\|c\|c\|}\hline 1&3\\\hline 2\\\cline{1-1} 4\\\cline{1-1}\end{array}$ $\begin{array}{\|c\|c\|}\hline 1&4\\\hline 2\\\cline{1-1} 3\\\cline{1-1}\end{array}$	3
2^2	$\begin{array}{\|c\|c\|}\hline 1&2\\\hline 3&4\\\hline\end{array}$ $\begin{array}{\|c\|c\|}\hline 1&3\\\hline 2&4\\\hline\end{array}$	2
$1^1 3^1$	$\begin{array}{\|c\|c\|c\|}\hline 1&2&3\\\hline 4\\\cline{1-1}\end{array}$ $\begin{array}{\|c\|c\|c\|}\hline 1&2&4\\\hline 3\\\cline{1-1}\end{array}$ $\begin{array}{\|c\|c\|c\|}\hline 1&3&4\\\hline 2\\\cline{1-1}\end{array}$	3
4^1	$\begin{array}{\|c\|c\|c\|c\|}\hline 1&2&3&4\\\hline\end{array}$	1

$$\sum_{\lambda} f_{\lambda}^2 = 1^1 + 3^2 + 2^2 + 3^2 + 1^1 = 24.$$

In general, $\sum_{\lambda} f_{\lambda}^2 = n!$. While this follows from certain results in representation theory, I'm not aware of an elementary proof.