Math 341 Homework 4 Solutions

Catalan numbers

1. Prove that for $n \geq 4$, the $(n-1)$–st Catalan number C_{n-1} is equal to the number of ways that a regular n-gon can be cut into triangles by connecting non-adjacent vertices by non-crossing line segments. (see picture below for C_4).

Solution. We will actually prove a slightly more general statement, which is easier to prove by induction: we will show that for each $n \geq 3$, there are C_{n-1} ways of cutting a convex n-gon into triangles by connecting non-adjacent vertices by non-crossing line segments (Recall that a subset of $\mathbb{R}^2$ is called convex if for every pair of points p, q in the set, the line segment joining p to q is contained in the set. In particular, every regular n-gon is convex).

We will prove the result by induction on n. If $n = 3$, then a triangle can be cut into triangles in $C_2 = 1$ one way (do nothing). Similarly, if $n = 4$ then a quadrilateral can be cut in $C_3 = 2$ ways into triangles by connecting non-adjacent vertices by non-crossing line segments. Now suppose $n \geq 4$, and the result has been proved for all values of $m \leq n$, and let P be a convex $(n+1)$-gon. Label the vertices of P by $v_1, v_2, \ldots, v_{n+1}$, where v_{n+1} is adjacent to v_n and v_1, and for $i = 2, \ldots, n$, v_i is adjacent to v_{i-1} and v_{i+1}.

Observe that if P is cut into triangles by connecting non-adjacent vertices by non-crossing line segments, then either (A): there is a line segment connecting v_2 and v_{n+1}, or (B): there is at least one line segment connecting v_1 to some vertex v_t, with $3 \leq t \leq n$.

If (A) occurs, then consider the convex n-gon determined by the vertices $v_2, v_3, \ldots, v_{n+1}$; by the induction hypothesis, there are C_{n-1} ways of cutting this n-gon into triangles by connecting non-adjacent vertices by non-crossing line segments. Thus there are C_{n-1} ways of cutting P into triangles by connecting non-adjacent vertices by non-crossing line segments so that option (A) occurs.

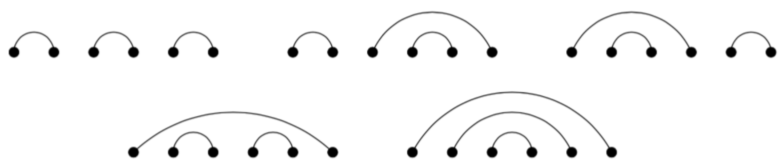
If (B) occurs, then let $2 \leq k \leq n-1$ be the largest integer so that there is a line segment connecting v_1 and v_{k+1}. Then the line joining v_1 and v_{k+1} cuts P into the convex $(k+1)$-gon determined by the vertices $v_1, \ldots, v_{k+1}$ and the convex $(n-k)$-gon determined by the vertices $v_{k+1}, v_{k+2}, \ldots, v_{n+1}, v_1$. By the induction hypothesis, there are C_k ways of cutting the $(k+1)$-gon into triangles, and C_{n-k} ways of cutting the $(n-k+1)$-gon into triangles. Summing over all possible k, we conclude that there are $\sum_{k=2}^{n-1} C_k C_{n-k}$ ways of cutting P into triangles by connecting non-adjacent vertices by non-crossing line segments so that option (B) occurs.

Thus all together, there are

$$C_{n-1} + \sum_{k=2}^{n-1} C_k C_{n-k} = \sum_{k=1}^{n-1} C_k C_{n-k} = C_n$$

ways of cutting P into triangles by connecting non-adjacent vertices by non-crossing line segments. This completes the induction.

2. Prove that C_n is equal to the number of non-crossing complete matchings on $2n-2$ vertices, i.e. the number of ways to connect $2n-2$ points in the plane, all lying on a horizontal line, using $n-1$ non-intersecting arcs, such that each arc connects two of the points, the arcs lie above the points, and no two arcs cross (see picture below for C_4).



Solution. First, observe that if $n = 2$, there is one non-crossing complete matching on $2n-2 = 2$ vertices. Now let $n > 2$ and suppose that for each $m = 2, \ldots, n-1$, we have proved that there are C_m non-crossing complete matchings on $2m-2$ vertices. We will now count the number of complete matchings on $2n-2$ vertices; label these vertices from left to right as $v_1, \ldots, v_{2n-2}$. Then for every complete matching on $v_1, \ldots, v_{2n-2}$, there is a unique number $p \geq 2$ so that v_1 is matched with v_p. If $p = 2n-2$, then there is a complete matching on the $2(n-1)-2 = 2((n-1)-1)$ vertices $v_2, v_3, \ldots, v_{2n-3}$. Thus there are C_{n-1} non-crossing complete matchings on $2n-2$ vertices so that v_1 is matched with v_{2n-2}.

If $p < 2n-2$, then we have a complete matching on the p vertices $v_1, \ldots, v_p$ and on the $2n-2-p$ vertices $v_{p+1}, \ldots, v_{2n-2}$. Observe that p must be even, i.e. $p = 2k-2$ for some integer $2 \leq k \leq n-1$. There are C_k complete matchings on the vertices $v_1, \ldots, v_{2k-2}$, and there are C_{n-k} complete matchings on the $2(n-k)-2$ vertices $v_{2k-1}, \ldots v_{2n-2}$. Thus all together, there are $\sum_{k=2}^{n-1} C_k C_{n-k}$ non-crossing complete matchings on $2n-2$ vertices so that v_1 is matched with a vertex other than v_{2n-2}.

Thus all together, there are

$$C_{n-1} + \sum_{k=2}^{n-1} C_k C_{n-k} = \sum_{k=1}^{n-1} C_k C_{n-k} = C_n$$

complete matchings on $2n-2$ vertices. This completes the induction.

3. A clown stands on the edge of a swimming pool, holding a bag containing n red and n blue balls. He draws the balls out one at a time (at random) and discards them. If he draws a blue ball, he takes one step back. If he draws a red ball, he takes one step forward (all steps have the same size). Prove that the probability that the clown remains dry is $1/(n+1)$.

Solution. There are $\binom{2n}{n}$ strings of letters R and B (red and blue) that contain exactly n R's and n B's. Each such string corresponds to a possible sequence or red and blue balls that the clown could draw from the bag. If any initial segment contains more R's than B's, it corresponds to the clown getting wet. Thus the probability that the clown gets wet is $x/\binom{2n}{n}$, where x is the number of strings containing n R's and n B's, where every initial segment contains at least as many B's as R's. We know that $x = C_{n+1}$, the $(n+1)$–st Catalan number (just replace B's with opening brackets and R's with closing brackets), and thus $n = \frac{1}{n+1}\binom{2n}{n}$. We conclude that the probability that the clown get wet is $x/\binom{2n}{n} = \frac{1}{n+1}$.

Equivalence relations

4. Let $S = \mathbb{R}\backslash\{0\}$ and define the relation $a \sim b$ if $a/b \in \mathbb{Q}$. Prove that this is an equivalence relation.

Solution. Let $a \in \mathbb{R}\backslash\{0\}$. Then $a/a = 1 \in \mathbb{Q}$, so $a \sim a$, i.e. the relation is reflexive. Next, suppose $a, b \in \mathbb{R}\backslash\{0\}$ with $a \sim b$. Then $a/b \in \mathbb{Q}$, i.e. we can write $a/b = p/q$, where p and q are non-zero integers (we know that $p \neq 0$ since $a \neq 0$). Thus $b/a = q/p \in \mathbb{Q}$, so $b \sim a$, i.e. the relation is symmetric. Finally, suppose $a \sim b$ and $b \sim c$. Then we can write $a/b = p/q$ and $b/c = r/s$, where p, q, r, s are non-zero integers. But then $a/c = (a/b)(b/c) = (p/q)(r/s) = (pr)/(rs) \in \mathbb{Q}$, so $a \sim c$, i.e. the relation is transitive. We conclude that $\sim$ is an equivalence relation.

5. A number $a \in \mathbb{R}$ is called *algebraic* if there is a nonzero polynomial $P(x)$ with integer coefficients (i.e. $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$, with $a_n, a_{n-1}, \ldots, a_0 \in \mathbb{Z}$ and $a_n \neq 0$) so that $P(a) = 0$. If $a \in \mathbb{R}$ is *not* algebraic, it is called *transcendental*.

Let $S = \mathbb{R}\backslash\{0\}$ and let $\sim$ be the equivalence relation from Problem 4. Let $a \in \mathbb{R}$ be a transcendental number. Prove that the equivalence classes $[[1]]$, $[[a]]$, $[[a^2]]$, $[[a^3]], \ldots$, are all distinct.

Solution. Suppose that $[[a^m]] = [[a^n]]$ for some integers $0 \leq m < n$. This means that $a^m \sim a^n$, so there is a rational number $\frac{p}{q}$ (with $q \neq 0$) so that $a^m = \frac{p}{q}a^n$, i.e. $qa^m - pa^n = 0$. Thus a is a root of the polynomial $P(x) = qx^m - px^n$, which is non-zero and has integer coefficients. This implies that a is algebraic. Since a is not algebraic, we must have $[[a^m]] \neq [[a^n]]$ whenever $0 \leq m < n$.

6. Let $S = \{1, 2, 3, 4\}$. Define $R = \{(1, 2), (2, 1), (1, 3),\ (3, 1),\ (3, 4), (4, 3)\}$. Is R an equivalence relation? Prove that your answer is correct.

Solution. No. We have $2 \sim 1$ and $1 \sim 3$, but $2 \not\sim 3$. Thus R fails to be transitive and thus is not an equivalence relation.

7. Let S be the set of all English words. Define an equivalence relation $a \sim b$ if the words a and b begin with the same letter. Is this an equivalence relation? Prove that your answer is correct.

Solution. Yes. First, if $a \in S$ is a word, then a begins with the same letter as a. Thus $a \sim a$, so the relation is reflexive. Next, if a and b are words so that b begins with the same letter as a, then a begins with the same letter as b. Thus $a \sim b$ implies $b \in a$, so the relation is symmetric. Finally, if a, b, and c are words so that b begins with the same letter as a and c begins with the same letter as b, then c begins with the same letter as a. Thus $a \sim b$ and $b \sim c$ implies $a \sim c$, so the relation is transitive. Thus $\sim$ is an equivalence relation.

Generating functions and permutations

8. Define the numbers $a_0, a_1, \ldots$ by

$$\prod_{m=1}^{\infty}(1 + t^m) = \sum_{n=0}^{\infty} a_n t^n.$$

Prove that a_n is the number of ways of writing n as a sum of *distinct* positive integers. E.g. $a_6 = 4$, since we can write $6 = 6,\ 6 = 5 + 1,\ 6 = 4 + 2,\ 6 = 3 + 2 + 1$.

Solution. One way to interpret the product $\prod_{m=1}^{\infty}(1 + t^m)$ is that for each term in the product $(1 + t)(1 + t^2)(1 + t^3)\cdots$, we must either choose the "1" term or the "t^m" term. Here's a way to make that statement precise. Let $\mathcal{F}$ be the set of all functions $f\colon \mathbb{N} \to \{0,1\}$. Then

$$\prod_{m=1}^{\infty}(1+t^m) = \sum_{f\in\mathcal{F}} t^{\sum_{k=1}^{\infty} kf(k)}.$$

Thus

$$a_n = \left|\left\{f \in \mathcal{F}\colon \sum_{k=1}^{\infty} kf(k) = n\right\}\right|.$$

But the set of functions $f \in \mathcal{F}$ that satisfies $\sum_{k=1}^{\infty} kf(k) = n$ is in one-to-one correspondence with the set of ways of writing n as a sum of distinct integers.

9. Solve the non-linear recurrence relation $f(n) = f(n-1)^2$, $f(0) = 2$. Hint: sometimes generating functions aren't the answer.

Solution. Lets compute the first few values of $f(n)$. We have $f(0) = 0$, $f(1) = f(0)^2 = 2^2 = 4$, $f(2) = f(1)^2 = 4^2 = 16$. This suggests that $f(n) = 2^{2^n}$ is a good guess. Lets check: $2^{2^0} = 2^1 = 2$. In general, $2^{2^n} = (2^{2^{n-1}})^2$. Thus the function $f(n) = 2^{2^n}$ satisfies $f(0) = 0$ and $f(n) = f(n-1)^2$.