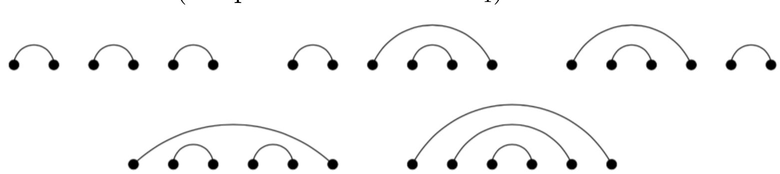
Math 341 Homework 4

- Due Thursday, March 15 at start of class.
- If your homework is longer than one page, **staple** the pages together, and put your name on each sheet of paper.
- You are allowed (and encouraged) to use results proved in class on your homework.
- **Collaboration Policy**: You are welcome (and encouraged) to work on the homework in groups. However, each student must write up the homework on their own, and must use their own wording (i.e. don't jusy copy the solutions from your friend). If you do collaborate with others, please list the name of your collaborators at the top of the homework.
- You are encouraged (though not required) to type up your solutions. If you choose to do this, I strongly recommend that you use the typesetting software LaTeX. LaTeX is used by the entire mathematics community, and if you intend to go into math, youll need to learn it sooner or later. "The Not So Short Introduction to LaTeX" is a good place to start. This guide can be found at http://tug.ctan.org/info/lshort/english/lshort.pdf . You can also download the .tex source file for this homework and take a look at that.
- Each homework problem should be correct as stated. Occasionally, however, I might screw something up and give you an impossible homework problem. If you believe a problem is incorrect, please email me. If you are right, the first person to point out an error will get +1 on that homework, and I will post an updated version.

Catalan numbers

1. Prove that for $n \geq 4$, the $(n-1)$–st Catalan number C_{n-1} is equal to the number of ways that a regular n-gon can be cut into triangles by connecting non-adjacent vertices by non-crossing line segments. (see picture below for C_4).

2. Prove that C_n is equal to the number of non-crossing complete matchings on $2n-2$ vertices, i.e. the number of ways to connect $2n-2$ points in the plane, all lying on a horizontal line, using $n-1$ non-intersecting arcs, such that each arc connects two of the points, the arcs lie above the points, and no two arcs cross (see picture below for C_4).



3. A clown stands on the edge of a swimming pool, holding a bag containing n red and n blue balls. He draws the balls out one at a time (at random) and discards them. If he draws a blue ball, he takes one step back. If he draws a red ball, he takes one step forward (all steps have the same size). Prove that the probability that the clown remains dry is $1/(n+1)$.

Equivalence relations

4. Let $S = \mathbb{R}\backslash\{0\}$ and define the relation $a \sim b$ if $a/b \in \mathbb{Q}$. Prove that this is an equivalence relation.

5. A number $a \in \mathbb{R}$ is called *algebraic* if there is a nonzero polynomial $P(x)$ with integer coefficients (i.e. $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_0$, with $a_n, a_{n-1}, \ldots, a_0 \in \mathbb{Z}$ and $a_n \neq 0$) so that $P(a) = 0$. If $a \in \mathbb{R}$ is *not* algebraic, it is called *transcendental*.

Let $S = \mathbb{R}\backslash\{0\}$ and let $\sim$ be the equivalence relation from Problem 4. Let $a \in \mathbb{R}$ be a transcendental number. Prove that the equivalence classes $[[1]]$, $[[a]]$, $[[a^2]]$, $[[a^3]], \ldots,$ are all distinct.

6. Let $S = \{1, 2, 3, 4\}$. Define $R = \{(1, 2), (2, 1), (1, 3),\ (3, 1),\ (3, 4), (4, 3)\}$. Is R an equivalence relation? Prove that your answer is correct.

7. Let S be the set of all English words. Define an equivalence relation $a \sim b$ if the words a and b begin with the same letter. Is this an equivalence relation? Prove that your answer is correct.

Generating functions and permutations

8. Define the numbers $a_0, a_1, \ldots$ by

$$\prod_{m=1}^{\infty}(1 + t^m) = \sum_{n=0}^{\infty} a_n t^n.$$

Prove that a_n is the number of ways of writing n as a sum of *distinct* positive integers. E.g. $a_6 = 4$, since we can write $6 = 6,\ 6 = 5 + 1,\ 6 = 4 + 2,\ 6 = 3 + 2 + 1$.

9. Solve the non-linear recurrence relation $f(n) = f(n-1)^2$, $f(0) = 2$. Hint: sometimes generating functions aren't the answer.