Fibonacci numbers

1. Prove that if \( n \) is divisible by three, then \( F_n \) is even; i.e., \( F_3, F_6, F_9, \text{ etc.} \) are even.

Solution. We will prove the result by induction. First, observe that \( F_3 = 2 \) is even. Now suppose that \( k \) is a positive integer and that \( F_{3k} \) is even. We have \( F_{3(k+1)} = F_{3k+2} + F_{3k+1} = 2F_{3k+1} + F_{3k} \).
Since \( 2F_{3k+1} \) is even and \( F_{3k} \) is even, and the sum of two even numbers is even, we conclude that \( F_{3(k+1)} \) is even. Thus \( F_{3k} \) is even for every positive integer \( k \). (Note that \( F_0 = F_{3\cdot 0} \) is also even).

2. Prove that if \( n \geq 6 \) and \( n \) is even, then \( F_n \) is composite (i.e. it is not prime). Hint: try expanding out the formula \( F_n = F_{n-1} + F_{n-2} \) multiple times.

Solution. Fix a positive integer \( n \geq 3 \). First, we will prove by induction that for every positive integer \( k \leq n - 1 \), \( F_n = F_{k+1}F_{n-k} + F_kF_{n-k-1} \). When \( k = 1 \), we have \( F_n = F_{n-1} + F_{n-2} = F_2F_{n-1} + F_1F_{n-2} \), so the base case holds. Now suppose the result has been proved for some \( k \leq n - 2 \). Then

\[
F_n = F_{k+1}F_{n-k} + F_kF_{n-k-1}
= F_{k+1}(F_{n-k-1} + F_{n-k-2}) + F_kF_{n-k-1}
= (F_{k+1} + F_k)F_{n-k-1} + F_{k+1}F_{n-k-2}
= F_{k+2}F_{n-k-1} + F_{k+1}F_{n-k-2}
= F_{(k+1)+1}F_{n-(k+1)} + F_{(k+1)}F_{n-(k+1)-1},
\]

which completes the induction step. Now suppose \( n \geq 6 \) is even, i.e. \( n = 2m \) for some positive integer \( m \). Applying the above result with \( k = m = n/2 \), we have

\[
F_n = F_{m+1}F_m + F_mF_{m-1} = F_m(F_{m+1} + F_{m-1}).
\]

Since \( n \geq 6 \), \( m \geq 3 \) and thus \( F_m \geq 2 \) and \( (F_{m+1} + F_{m-1}) \geq 2 \), i.e. \( F_m \) can be written as the product of two integers, each of which is \( \geq 2 \). Thus \( F_n \) is composite.

Generating functions

3. Expand \( \frac{2t}{1-8t+15t^2} \) as a power series (i.e. write it in the form \( \sum_{n \geq 0} a_n t^n \), and compute the numbers \( a_n \)).

Solution. We have \( 1-8t+15t^2 = (1-5t)(1-3t) \). Thus we can compute a partial fraction expansion:

\[
\frac{2t}{1-8t+15t^2} = \frac{A}{1-5t} + \frac{B}{1-3t}.
\]

Solving \( A(1-3t) + B(1-5t) = 2t \), we see that \( A = 1 \) and \( B = -1 \), i.e.

\[
\frac{2t}{1-8t+15t^2} = \frac{1}{1-5t} - \frac{1}{1-3t}.
\]
Thus the power series expansion for \( \frac{2t}{1-8t+15t^2} \) is
\[
\sum_{n=0}^{\infty} 5^n t^n - \sum_{n=0}^{\infty} 3^n t^n = \sum_{n=0}^{\infty} (5^n - 3^n) t^n.
\]

4. Consider the sequence \( a_n \) defined by \( a_0 = 0 \) and \( a_{n+1} = 3a_n + 2 \) for \( n \geq 0 \). Using the method of generating functions, write down a formula for \( a_n \).

Solution. Define \( \phi(t) = \sum_{n=0}^{\infty} a_n t^n \). We have
\[
\phi(t) = \sum_{n=1}^{\infty} a_n t^n = \sum_{n=1}^{\infty} (3a_{n-1} + 2)t^n \\
= 3t \sum_{n=0}^{\infty} a_n t^n + 2 \sum_{n=0}^{\infty} t^n - 2 \\
= 3t \phi(t) + \frac{2}{1-t} - 2,
\]
so \( \phi(t)(1 - 3t) = \frac{2}{1-t} - 2 \), i.e. \( \phi(t) = \frac{2}{(1-t)(1-3t)} - \frac{2}{1-3t} \). Taking a partial fraction expansion of
\[
\frac{2}{(1-t)(1-3t)} = \frac{-1}{1-t} + \frac{3}{1-3t},
\]
we obtain
\[
\phi(t) = -1 + \frac{3}{1-3t} - \frac{2}{1-3t} \\
= -1 + \frac{1}{1-3t} \\
= \sum_{n=0}^{\infty} (3^n - 1)t^n
\]
Thus \( a_n = 3^n - 1 \) for each integer \( n \geq 0 \).

5. Consider the sequence \( a_n \) defined by \( a_0 = 0 \), \( a_1 = 1 \), and \( a_{n+2} = 2a_{n+1} - a_n \) for \( n \geq 0 \). Using the method of generating functions, write down a formula for \( a_n \).

Solution. Define \( \phi(t) = \sum_{n=0}^{\infty} a_n t^n \). We have
\[
\phi(t) = \sum_{n=1}^{\infty} a_n t^n = t \sum_{n=2}^{\infty} a_n t^n \\
= t + \sum_{n=2}^{\infty} (2a_{n-1} - a_{n-2})t^n \\
= t + 2t \sum_{n=0}^{\infty} a_n t^n - t^2 \sum_{n=0}^{\infty} a_n t^n \\
= t + 2t \phi(t) - t^2 \phi(t),
\]
so \( \phi(t)(1-2t+t^2) = t \), or

\[
\phi(t) = \frac{t}{1-2t+t^2} = \frac{t}{(1-t)^2}.
\]

Since the denominator has a repeated root, we can't use the method of partial fractions to write

\[
\frac{t}{(1-t)^2} = \frac{A}{1-t} + \frac{B}{(1-t)^2}.
\]

However, we can compute directly the series expansion of \( \frac{t}{(1-t)^2} \). Let's take a few derivatives:

\[
\phi'(t) = \frac{1+t}{(1-t)^3}, \quad \phi''(t) = \frac{2(2+t)}{(1-t)^4}, \quad \phi'''(t) = \frac{6(3+t)}{(1-t)^5}.
\]

We'll prove by induction that \( \phi^{(k)}(t) = \frac{k!(k+t)}{(1-t)^{k+2}} \). We've already established the base case \( k = 1 \) (and also \( k = 2 \) and \( k = 3 \) for that matter). Now for the induction step:

\[
\phi^{(k+1)}(t) = \left( \frac{k!(k+t)}{(1-t)^{k+2}} \right)'
= k!(k+t)(-(k+2)(1-t)^{k+1}) - (1-t)^{k+2}
= k!(k+1)(1-t)^{k+2}((k+1)+t)
= (k+1)!(1-t)^{(k+1)+2},
\]

which completes the induction. Thus the function \( \phi(t) \) has the expansion \( \phi(t) = \sum_{n=0}^{\infty} b_n t^n \), where

\[
b_n = \frac{1}{n!} \phi^{(n)}(0) = \frac{1}{n!} \frac{n!(n+0)}{(1-0)^{n+2}} = n.
\]

We conclude that \( a_n = n \) for each \( n \geq 0 \).

We can also verify this directly, since \( n + 2 = 2(n + 1) \) (perhaps the moral of the story is that sometimes having a lucky guess is easier than using generating functions).

**Permutations**

6. Let \( \pi \in S_n \). Prove the formula \( \text{sgn}(\pi) = (-1)^{n-C(\pi)-F(\pi)} \) from lecture.

**Solution.** Recall that if \( (a_1 \ a_2 \ \ldots \ a_k) \) is a cycle, then we can write

\[
(a_1 \ a_2 \ \ldots \ a_k) = (a_1 \ a_2)(a_1 \ a_3)(a_1 \ a_4) \cdots (a_1 \ a_k).
\]

The right hand side of the above equation is a product of \( k-1 \) transpositions, and thus its sign is \( (-1)^{k-1} \). Now suppose \( \pi \in S_n \), and suppose \( \pi \) can be written as a product of disjoint cycles of lengths \( k_1, k_2, \ldots, k_t \) \( t = C(\pi) \). We have \( n = k_1 + k_2 + \ldots + k_t + F(\pi) \), and

\[
\text{sgn}(\pi) = (-1)^{k_1-1}(-1)^{k_2-1} \cdots (-1)^{k_t-1}
= (-1)^{k_1+k_2+\ldots+k_t}(-1)^{-t}
= (-1)^{n-F(\pi)-t}
= (-1)^{n-C(\pi)-F(\pi)}.
\]

7. a. Let \( \pi \in S_n \) be of the form \( \pi = (a_1 \ a_2 \ \ldots \ a_k) \). Prove that \( \pi \) is even if and only if \( k \) is odd. For part b, we will call \( k \) the “length” of the cycle.

**Solution.** As noted in problem 6, we can write \( \pi = (a_1 \ a_2 \ \ldots \ a_k) = (a_1 \ a_2)(a_1 \ a_3)(a_1 \ a_4) \cdots (a_1 \ a_k) \).

This is a product of \( k-1 \) transpositions. Thus \( \pi \) is even if and only if \( k \) is odd.
b. Let \( \pi \in S_n \), and consider a representation of \( \pi \) as a product of disjoint cycles. Prove that \( \pi \) is even if and only if there are an even number of even length cycles (and any number of odd length cycles).

Solution. Write \( \pi \) as a product of disjoint cycles of lengths \( k_1, k_2, \ldots, k_t, t = C(\pi) \). Then sgn(\( \pi \)) = \((-1)^{k_1-1}(-1)^{k_2-1}\cdots(-1)^{k_t-1} \). If \( k_j \) is odd then \( k_j - 1 \) is even, so \((-1)^{k_j-1} = 1\). Conversely, if \( k_j \) is even, then \( k_j - 1 \) is odd, so \((-1)^{k_j-1} = -1\). Thus sgn(\( \pi \)) = -1 if and only if there are an odd number of cycles of even length.