Math 341 Homework 2

- Due Thursday, February 8 at start of class.
- If your homework is longer than one page, **staple** the pages together, and put your name on each sheet of paper.
- You are allowed (and encouraged) to use results proved in class on your homework.
- **Collaboration Policy**: You are welcome (and encouraged) to work on the homework in groups. However, each student must write up the homework on their own, and must use their own wording (i.e. don't jusy copy the solutions from your friend). If you do collaborate with others, please list the name of your collaborators at the top of the homework.
- You are encouraged (though not required) to type up your solutions. If you choose to do this, I strongly recommend that you use the typesetting software LaTeX. LaTeX is used by the entire mathematics community, and if you intend to go into math, youll need to learn it sooner or later. "The Not So Short Introduction to LaTeX" is a good place to start. This guide can be found at http://tug.ctan.org/info/lshort/english/lshort.pdf . You can also download the .tex source file for this homework and take a look at that.
- Each homework problem should be correct as stated. Occasionally, however, I might screw something up and give you an impossible homework problem. If you believe a problem is incorrect, please email me. If you are right, the first person to point out an error will get +1 on that homework, and I will post an updated version.

1. Let m and n be integers and let a be a positive integer. We say $m \equiv n \pmod a$ if there is an integer t so that $m-n = ta$ (i.e. $m-n$ is divisible by a). If the statement "$m \equiv n \pmod a$" is false, we write $m \not\equiv n \pmod a$. HW1 problem 4 implies that if p is prime, then $(1+n)^p \equiv 1+n^p \pmod p$ for every positive integer n. You may use this fact to solve the problem below.
a) Using induction, prove that if p is a prime, then $n^p \equiv n \pmod p$ for every positive integer n.
b) Give an example of positive integers n and a so that $n^a \not\equiv n \pmod a$.

2. Prove that

$$\binom{2n}{n} = \frac{2^{2n}}{\sqrt{\pi n}}\Big(1 + O(\frac{1}{n})\Big).$$

(Here $\pi = 3.141...$ is a real number, not a permutation!)
Update January 31: This problem seems a bit hard. So I've decided to make it a bonus problem (if you solve it, it is possible to get above 100% on this homework). Furthermore, here are some hints. As a first step, it might be useful to consider the function $f(n) = n! - \sqrt{2\pi n}(n/e)^n$. One reason this function is useful is that there exists an absolute constant C so that

$$\Big|\frac{f(n)}{\sqrt{2\pi n}(n/e)^n}\Big| \leq \frac{C}{n}\sqrt{2\pi n}(n/e)^n,$$

and

$$\Big|\frac{f(n)}{n!}\Big| \leq \frac{C}{n}\sqrt{2\pi n}(n/e)^n.$$

These two estimates are easy to prove, but you can use them without proof.

A second hint is that $(n!)^2 \leq (2n)!$ and $(2n)! \leq 2^{2n}(n!)^2$ (this was updated on February 5, 2018; the previous (incorrect) inequality was $(2n)! \leq 2^n(n!)^2$. If you already wrote your solution using the old inequality, you can leave it unchanged) Again, this is easy to prove but you can use these results without providing a proof.

3. Let n be a positive integer. Prove that for every $\pi \in S_n$, there is a positive integer t so that $\pi^t = e$, where e is the identity permutation.

(Here π is a permutation, not the real number 3.141...!)

Food for thought: If n is fixed, what is the largest t can be? Can you come up with an interesting bound on the size of t?

Vectors and the symmetric group

For the next set of questions, we will have the following setup.

Let n be a positive integer and let $v_1, \ldots, v_n$ be vectors in $\mathbb{R}^n$. We'll use the notation

$$\begin{aligned} v_1 &= (v_{1,1}, v_{1,2}, \ldots, v_{1,n}), \\ v_2 &= (v_{2,1}, v_{2,2}, \ldots, v_{2,n}), \\ &\vdots \\ v_n &= (v_{n,1}, v_{n,2}, \ldots, v_{n,n}). \end{aligned}$$

Consider the following function, whose input is n vectors and whose output is a real number:

$$f(v_1, \ldots, v_n) = \sum_{\pi \in S_n} \operatorname{sgn}(\pi) \prod_{j=1}^{n} v_{j\pi,j}. \tag{1}$$

(Recall our notation from class that if $\pi \in S_n$ and $j \in [n]$, then $j\pi$ is the integer obtained by applying the permutation π to the integer j. We will define $\operatorname{sgn}(\pi)$ in class on Jan 30.)

For example, if $n = 2$ then S_2 is the set containing the permutations $\pi_1 = e$ and $\pi_2 = (1, 2)$. We have $\operatorname{sgn}(\pi_1) = 1$ and $\operatorname{sgn}(\pi_2) = -1$. Thus, if $v_1 = (v_{1,1}, v_{1,2})$ and $v_2 = (v_{2,1}, v_{2,2})$, then

$$f(v_1, v_2) = v_{1,1}v_{2,2} - v_{2,1}v_{1,2}. \tag{2}$$

4. Let $n = 3$ and let $v_1 = (v_{1,1}, v_{1,2}, v_{1,3})$ and similarly for v_2 and v_3. Write down $f(v_1, v_2, v_3)$ explicitly as a sum of products of the numbers $v_{i,j}$, in the same style as Equation (2) above. Compare this with the formula for the determinant

$$\begin{vmatrix} v_{1,1} & v_{1,2} & v_{1,3} \\ v_{2,1} & v_{2,2} & v_{2,3} \\ v_{3,1} & v_{3,2} & v_{3,3} \end{vmatrix}$$

Are the two expressions the same?

5. Let $v_1 = (1, \ldots, 0)$, $v_2 = (0, 1, 0, \ldots, 0)$, $v_3 = (0, 0, 1, \ldots, 0)$, and in general let v_j be the vector that has a one in the j–th position and zeroes elsewhere. Prove that

$$f(v_1, \ldots, v_n) = 1.$$

6. Prove that if two of the vectors are interchanged, then the sign of f flips. More formally, prove that if $v_1, \dots, v_n$ are vectors in $\mathbb{R}^n$, then

$$f(v_1, \dots, v_i, \dots, v_j, \dots, v_n) = -f(v_1, \dots, v_j, \dots, v_i, \dots, v_n).$$

(in the above expression, each of the vectors $v_1, \dots, v_n$ are in the same place, except the vectors v_i and v_j have switched places on the right hand side).

Hint: it might be helpful to recall the definition of $\text{sgn}(\pi)$.

7. Let $v_1, \dots, v_n$ be vectors in $\mathbb{R}^n$. Suppose that two of the vectors are the same, i.e. $v_i = v_j$ for some $i \neq j$. Prove that

$$f(v_1, \dots, v_n) = 0.$$

8. Let $v_1, \dots, v_n$ be vectors in $\mathbb{R}^n$, let $1 \leq j \leq n$ and let v'_j be a vector in $\mathbb{R}^n$. Prove that

$$f(v_1, \dots, v_j + v'_j, \dots, v_n) = f(v_1, \dots, v_j, \dots, v_n) + f(v_1, \dots, v'_j, \dots, v_n).$$