Math 341 Homework 1 Solutions

All problems are worth 5 points, for a total of 45.

Sets

For these problems, we will need some notation not discussed in class. If S and T are sets, $S \cup T$ is the set containing all elements that are in S or in T (or in both). This is called the *union* of S and T. Similarly, $S \cap T$ is the set containing all elements that are in S and in T. This is called the *intersection* of S and T.

We can extend this definition to take a union of multiple sets. For example, if $S_1, \ldots S_k$ are sets, we write $S_1 \cup S_2 \cup \ldots \cup S_k$, or $\bigcup_{i=1}^k S_i$ to denote the union of $S_1, \ldots, S_k$; this is the set of elements that are contained in at least one of the sets $S_1, \ldots, S_k$. Similarly, we write $S_1 \cap S_2 \cap \ldots \cap S_k = \bigcap_{i=1}^k S_i$ to denote the intersection of $S_1, \ldots, S_k$; this is the set of elements that are contained in all of the sets $S_1, \ldots, S_k$.

1. Let S and T be sets of finite cardinality (i.e. both $|S|$ and $|T|$ are finite). Prove that

$$|S \cup T| = |S| + |T| - |S \cap T|.$$

Solution First, it follows immediately from the definition of cardinality that if A and B are finite sets with $A \cap B = \emptyset$, then $|A \cup B| = |A| + |B|$. Similarly, if A, B, C are finite sets, no two of which contain an element in common, then $|A \cup B \cup C| = |A| + |B| + |C|$. In the other extreme, if A and B are finite sets with $A \subset B$, then $|B \backslash A| = |B| - |A|$; here $B \backslash A = \{b \in B \colon b \notin A\}$. Observe as well that $A \cap B \subset A$ and $A \cap B \subset B$.

Observe that we can write $T = (T \backslash S) \cup (T \cap S)$. Since these two sets are disjoint, we have $|T| = |T \backslash S| + |S \cap T|$. Similarly, $|S| = |S \backslash T| + |S \cap T|$.

Now, observe that the three we can write $S \cup T = (S \backslash T) \cup (T \backslash S) \cup (S \cap T)$, and these three sets are disjoint. Thus

$$|S \cup T| = |S \backslash T| + |T \backslash S| + |S \cap T| = \big(|S| - |S \cap T|\big) + \big(|T| - |S \cap T|\big) + |S \cap T| = |S| + |T| - |S \cap T|.$$

2. Let S be a nonempty set. Without using the binomial theorem, prove that S has the same number of subsets of even and of odd cardinality. Hint: it might be useful to consider the cases where $|S|$ is even and where $|S|$ is odd separately.

Solution. It is tempting to just use the bijection $f \colon \mathcal{P}(S) \to \mathcal{P}(S)$ given by $f(A) = S \backslash A$. While this works when $|S|$ is odd, it does not work when $|S|$ is even. Instead, we will need a slightly more involved proof.

Let $\mathcal{S}$ be a set of sets. Define

$$\mathcal{O}(\mathcal{S}) = |\{A \in \mathcal{S} \colon A \text{ has odd cardinality}\},$$
$$\mathcal{E}(\mathcal{S}) = |\{A \in \mathcal{S} \colon A \text{ has even cardinality}\}.$$

Our goal is to show that for every set S of cardinality at least one, $\mathcal{O}(\mathcal{P}(S)) = \mathcal{E}(\mathcal{P}(S))$. Let S be a set of cardinality at least one. Select an element $a \in S$. Define

$$\begin{aligned}
\mathcal{S}_1 &= |\{A \in \mathcal{P}(S) \colon |A| \text{ is even}, \ a \in A\}, \\
\mathcal{S}_2 &= |\{A \in \mathcal{P}(S) \colon |A| \text{ is odd}, \ a \in A\}, \\
\mathcal{S}_3 &= |\{A \in \mathcal{P}(S) \colon |A| \text{ is even}, \ a \notin A\}, \\
\mathcal{S}_4 &= |\{A \in \mathcal{P}(S) \colon |A| \text{ is odd}, \ a \notin A\}.
\end{aligned}$$

We have that $\mathcal{P}(S) = A_1 \cup A_2 \cup A_3 \cup A_4$, and these four sets are disjoint, i.e. $A_i \cap A_j = \emptyset$ if $i \neq j$. Thus $|\mathcal{O}(\mathcal{P}(S))| = |\mathcal{S}_2| + |\mathcal{S}_4|$ and $|\mathcal{E}(\mathcal{P}(S))| = |\mathcal{S}_1| + |\mathcal{S}_3|$.

Define $S' = S \backslash \{a\}$. Observe that there is a bijection between $\mathcal{S}_1$ and $\mathcal{O}(\mathcal{P}(S'))$, given by $A \mapsto A\backslash\{a\}$. In words, if $A \in \mathcal{S}_1$, then $a \in A$ and $|A|$ is even; thus $A\backslash\{a\} \in \mathcal{P}(A')$ and $|A\backslash\{a\}|$ is odd, and conversely. We conclude that

$$\mathcal{S}_1 = \mathcal{O}(\mathcal{P}(A')),$$

and similarly,

$$\mathcal{S}_2 = \mathcal{E}(\mathcal{P}(A')).$$

Next, observe that there is a bijection between $\mathcal{S}_3$ and $\mathcal{E}(\mathcal{P}(A'))$ given by $A \mapsto A$. In words, if $A \in \mathcal{S}_3$, then $a \notin A$ and $|A|$ is even; thus $A \in \mathcal{P}(A')$, and conversely. We conclude that

$$\mathcal{S}_3 = \mathcal{E}(\mathcal{P}(A')),$$

and similarly,

$$\mathcal{S}_4 = \mathcal{O}(\mathcal{P}(A')).$$

Thus

$$\begin{aligned}
|\mathcal{O}(\mathcal{P}(S))| &= |\mathcal{E}(\mathcal{P}(A'))| + |\mathcal{O}(\mathcal{P}(A'))|, \\
|\mathcal{E}(\mathcal{P}(S))| &= |\mathcal{O}(\mathcal{P}(A'))| + |\mathcal{E}(\mathcal{P}(A'))|,
\end{aligned}$$

which implies $|\mathcal{O}(\mathcal{P}(S))| = |\mathcal{E}(\mathcal{P}(S))|$.

3. Let k and n be positive integers. Let $A_1, \ldots A_k$ be sets. Suppose that

$$\Big| \bigcup_{i=1}^{k} A_i \Big| = n^2,$$

that

$$|A_i| \geq 2n \text{ for every } 1 \leq i \leq k,$$

and that

$$|A_i \cap A_j| \leq 1 \quad \text{for every } 1 \leq i < j \leq k.$$

Prove that $k \leq n$.

Solution. We will prove the result by contradiction. Suppose $k > n$. For each $i = 1, \ldots, n$, define

$$A_i' = A_i \backslash \bigcup_{\substack{j=1,\ldots,n \\ j \neq i}} A_j = A_i \backslash (A_1 \cup A_2 \cup \ldots, \cup A_{i-1} \cup A_{i+1} \cup \ldots \cup A_n).$$

Since $|A_i \cap A_j| \leq 1$ for each index $j \neq i$, we have $|A_i'| \geq |A_i| - (n-1) > n$. Furthermore, the sets $A_1', \dots A_n'$ are disjoint. Thus

$$n^2 = \Big|\bigcup_{i=1}^{k} A_i\Big| \geq \Big|\bigcup_{i=1}^{n} A_i\Big| \geq \Big|\bigcup_{i=1}^{n} A_i'\Big| = \sum_{i=1}^{n} |A_i'| > \sum_{i=1}^{n} n = n^2,$$

which is a contradiction.

Binomial theorem

4. Let p be a prime number. Prove that there exists a polynomial $Q(x)$ with positive integer coefficients so that

$$(1+x)^p - (1+x^p) = pQ(x).$$

Solution. First, we will prove that if p is a prime and if $1 \leq k < p$ is an integer, then p does not divide $k!$. For each prime p, we will prove this by induction on k. If $k = 1$ then $k! = 1$ and p clearly does not divide k (since p prime implies $p \geq 2$). Next suppose that $1 \leq k < p$ and that we have already proved p does not divide $(k-1)!$. If p divides $k!$ then either p divides k or p divides $(k-1)!$. By the induction hypothesis, p does not divide $(k-1)!$. But since $k < p$, p cannot divide k. We conclude that p does not divide $k!$, which completes the induction step.

We are now ready to solve the problem. Using the binomial theorem, we have

$$(1+x)^p - (1+x^p) = \sum_{k=1}^{p-1} \binom{p}{k} x^k.$$

Recall that for $0 \leq k \leq p$, $\binom{p}{k} = \frac{p!}{k!(p-k)!}$. Since $k < p$ and $p - k < p$, we know from above that p does not divide $k!$, nor does it divide $(p-k)!$. Thus it does not divide $k!(p-k)!$. On the other hand, $p! = p(p-1)!$, so p *does* divide $p!$. We conclude that p divides $\binom{p}{k}$, i.e. we can write $\binom{p}{k} = pa_k$, where a_k is a positive integer (specifically, $a_k = \frac{(p-1)!}{k!(p-k)!}$).

We can now re-write the above equation as

$$(1+x)^p - (1+x^p) = p\sum_{k=1}^{p-1} a_k x^k = pQ(x),$$

where $Q(x) = \sum_{k=1}^{p-1} a_k x^k$.

5. Prove that for every positive integer n and every integer k, we have

$$k\binom{n}{k} = n\binom{n-1}{k-1}.$$

Hint: there's a reason this is is in the "Binomial Theorem" section.

Solution. Recall the binomial theorem

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k.$$

This is an identity about polynomials. Since polynomials are differentiable, we can differentiate both sides of this equation to obtain

$$n(1+x)^{n-1} = \sum_{k=1}^{n} \binom{n}{k} k x^{k-1}.$$

Next, apply the binomial theorem again to obtain

$$n(1+x)^{n-1} = n \sum_{k=0}^{n-1} \binom{n-1}{k} x^k = \sum_{k=0}^{n-1} n \binom{n-1}{k} x^k = \sum_{k=1}^{n} n \binom{n-1}{k-1} x^{k-1},$$

i.e.

$$\sum_{k=1}^{n} k \binom{n}{k} x^{k-1} = \sum_{k=1}^{n} n \binom{n-1}{k-1} x^{k-1}.$$

Since these two polynomials are equal, the coefficients of each term must be equal. In particular, we must have $k\binom{n}{k} = n\binom{n-1}{k-1}$.

Alternately, the result could also be proved directly by doing some algebra.

The size of $n!$

6. Recall that a polynomial is a function of the form $P(n) = a_C n^C + a_{C-1} n^{C-1} + \ldots + a_0$, where $C \geq 0$ is an integer and $a_0, \ldots, a_C$ are real numbers.

Prove that for every polynomial P, there exists a number N so that $n! > P(n)$ for all $n \geq N$.

Solution. Recall what we prove in lecture: For every positive integer C, there is a number N so that $n! \geq n^C$ for all $n \geq N$. This fact will be useful for us later on. Let $P = a_C n^C + a_{C-1} n^{C-1} + \ldots + a_0$ be a polynomial. Define $A = |a_C| + |a_{C-1}| + \ldots |a_0|$. Note that if $n \geq A+1$, then

$$\begin{aligned}
n^{C+1} &= n n^C \\
&> A n^C \\
&= (|a_C| + |a_{C-1}| + \ldots + |a_0|) n^C \\
&= |a_C| n^C + |a_{C-1}| n^C + \ldots + |a_0| n^C \\
&\geq |a_{C-1}| n^{C-1} + |a_{C-2}| n^{C-2} + \ldots + |a_0| \\
&\geq |a_C n^C + a_{C-1} n^{C-1} + \ldots + a_0| \\
&= |P(n)| \\
&\geq P(n).
\end{aligned}$$

Apply the result mentioned above to $C+1$, and let N_1 be the resulting number, i.e. for all $n \geq N_1$, we have $n! \geq n^{C+1}$. Define $N = \max(A+1, N_1)$. Then for all $n \geq N$, we have

$$n! \geq n^{C+1} > P(n),$$

as desired.

7. Prove that for every real number $C > 1$, there exists a number N so that $n! > C^n$ for all $n \geq N$.
Define N to be the smallest integer larger than $2C^2$. First we will suppose that n is even, so $n/2$

is an integer. Observe that if $n \geq N$, then for each $j = 0, 1, \ldots n/2$, $n - j \geq n/2 \geq C^2$. Thus for each $n \geq N$, we have

$$\begin{aligned} n! &= n(n-1)(n-2)\ldots 1 \\ &\geq n(n-1)\ldots(n/2) \\ &\geq (C^2)(C^2)\cdots(C^2) \\ &= (C^2)^{n/2+1} \\ &= C^{n+2} \\ &> C^n. \end{aligned}$$

If n is odd, then a similar argument shows that

$$\begin{aligned} n! &= n(n-1)(n-2)\ldots 1 \\ &\geq n(n-1)\ldots((n+1)/2)((n-1)/2) \\ &\geq (C^2)(C^2)\cdots(C^2)((n-2)/2) \\ &= (C^2)^{n/2}((n-2)/2) \\ &> C^n. \end{aligned}$$

Statistical physics and entropy

8. In this problem we will investigate the entropy of two different gases in the same box using the lattice model discussed in class.

Consider the system consisting of k_1 oxygen molecules and k_2 nitrogen molecules in a box. The box contains n possible locations where a gas molecule can reside, and at most one molecule can be located in each location. All oxygen molecules are indistinguishable, and all nitrogen molecules are indistinguishable, but oxygen and nitrogen molecules can be distinguished from each other.

What is the entropy of this system? Prove that your answer is correct.

Solution. Using the entropy formula, the entropy of this system is $k_B \log W$, where W is the number of microstates. Thus our task is to calculate W. If each oxygen molecule and each nitrogen molecule had a label, then the number of microstates would be $n(n-1)\cdots(n-k_1-k_2+1) = n!/(n-k_1-k_2)!$. However, since oxygen molecules are indistinguishable from each other and nitrogen molecules are indistinguishable from each other, the number of microstates is reduced by a multiplicative factor of $k_1!k_2!$, i.e.

$$W = \frac{n!}{(n-k_1-k_1)!k_1!k_2!}.$$

Students are also welcome to simplify this formula using Stirling's approximation and the parameters θ_1 and θ_2, which measure the concentration of oxygen and nitrogen molecules.

9. Consider a system consisting of k oxygen molecules in a box. The box contains n possible locations. Will adding an additional oxygen molecule to the box increase the entropy of the system? Prove that your answer is correct. (Hint: the answer may depend on the value of k and n).

Solution. If $k = n$ then it is not possible to add another molecule to the box. For $0 \leq k < n$, let W_1 be the number of microstates before the molecule is added, and let W_2 be the number of microstates after the molecule is added. We have

$$\frac{W_1}{W_2} = \frac{\binom{n}{k+1}}{\binom{n}{k}} = \frac{k!(n-k)!}{(k+1)!(n-k-1)!} = \frac{n-k}{k+1}.$$

Observe that $W_1/W_2 > 1$ if $n-k > k+1$ (i.e. $k < (n-1)/2$), while $W_1/W_2 < 1$ if $n-k < k+1$ (i.e. $k > (n-1)/2$). If $k = (n-1)/2$ then $W_1/W_2 = 1$.

We wish to determine whether adding another molecule increases or decreases entropy, i.e. we need to determine whether $k_B \log(W_1)$ is larger or smaller than $k_B \log(W_2)$. But $k_B \log W_1 - k_B \log W_2 = k_B \log(W_1/W_2)$. Since k_B is positive, we have that $k_B \log W_1 - k_B \log W_2$ if and only if $W_1/W_2 > 1$, i.e. if and only if $k < (n-1)/2$. Similarly, $k_B \log W_1 - k_B \log W_2 < 0$ if and only if $k > (n-1)/2$, and $k_B \log W_1 - k_B \log W_2 = 0$ if and only if $k = (n-1)/2$.

We conclude: If $0 \leq k < (n-1)/2$, then adding another molecule increases entropy. If If $(n-1)/2 < k < n$, then adding another molecule decreases entropy. If $k = (n-1)/2$ then adding another molecule leaves the entropy unchanged.