1. 
   a) Define 
   \[ f(x) = \begin{cases} 
   x^2, & x \neq 0, \\
   1, & x = 0
   \end{cases} \]
   What type of discontinuity does \( f \) have at 0? You do not need to prove that your answer is correct.
   \textit{Solution}. Removable discontinuity.
   
   b) Define 
   \[ f(x) = \begin{cases} 
   x^2, & x \geq 0, \\
   1 - x^2, & x < 0
   \end{cases} \]
   What type of discontinuity does \( f \) have at 0? You do not need to prove that your answer is correct.
   \textit{Solution}. Jump discontinuity
   
   c) Define \( f(x) = 1/x^2 \). What type of discontinuity does \( f \) have at 0? You do not need to prove that your answer is correct.
   \textit{Solution}. Infinite discontinuity
   
   d) Define \( f(x) = x^2 \). What type of discontinuity does \( f \) have at 0? You do not need to prove that your answer is correct.
   \textit{Solution}. No discontinuity! \( f \) is continuous at 0.
2. (10 points) Recall that a function $f$ is called injective (or one-to-one) if for all $x, y \in D(f)$ with $x \neq y$, we have $f(x) \neq f(y)$.

a) Let $f$ be a function that is differentiable at every point $x \in \mathbb{R}$, and suppose that $f'(x) > 0$ for all $x \in \mathbb{R}$. Must it be true that $f$ is injective? If yes, then prove it. If not, then give an example of a function $f$ for which the statement is not true, and show that your example is correct.

Solution. Yes, $f$ must be injective. Suppose instead that there existed distinct points $a, b \in \mathbb{R}$ with $f(a) = f(b)$. Without loss of generality, we can assume that $a < b$. Since $f$ is differentiable on $\mathbb{R}$, it is differentiable on $(a, b)$. Since $f$ is differentiable on $\mathbb{R}$, it is continuous on $\mathbb{R}$, and thus continuous on $[a, b]$. Thus we can apply Rolle’s theorem to conclude that there exists a point $x \in (a, b)$ with $f'(x) = 0$. But this contradicts the fact that $f'(x) > 0$ for all $x \in \mathbb{R}$.

b) Let $f$ be a function that is differentiable at every point $x \in \mathbb{R}$, and suppose that $f'(x) \geq 0$ for all $x \in \mathbb{R}$. Must it be true that $f$ is injective? If yes, then prove it. If not, then give an example of a function $f$ for which the statement is not true, and show that your example is correct.

Solution. No, $f$ need not be injective. For example, consider $f(x) = 0$. Then $f$ is differentiable at every point $x \in \mathbb{R}$, and $f'(x) = 0$ for all $x \in \mathbb{R}$, hence $f'(x) \geq 0$ for all $x \in \mathbb{R}$. However, $f(0) = 0 = f(1)$, so $f$ is not injective.
3. (10 points). Use Newton’s method to approximate the positive square root of seven, i.e. \( \sqrt{7} \).

To do this, write down a function with integer coefficients for which \( \sqrt{7} \) is a root, choose an appropriate starting guess \( x_0 \), and then iterate twice to determine \( x_1 \) and \( x_2 \). You do not need to simplify your fractions. You will be evaluated based on your choice of function, your choice of starting point, and the correct application of Newton’s method.

**Solution.** Let \( f(x) = x^2 - 7 \). Since \( 2^2 = 4, \ 3^3 = 9 \), and \( f \) is continuous, by the intermediate value theorem there exists a point \( 2 < x < 3 \) with \( f(x) = 0 \). Thus both 2 and 3 are good candidates for our first guess of Newton iteration. Let \( x_0 = 2 \). We have

\[
\Delta_0 = \frac{f(2)}{f'(2)} = \frac{-3}{4},
\]

so \( x_1 = x_0 - \Delta_0 = 2 + 3/4 = 11/4 \).

We have

\[
\Delta_1 = \frac{f(11/4)}{f'(11/4)} = \frac{121/16 - 7}{22/4} = \frac{9/16}{22/4} = 9/88,
\]

so \( x_2 = x_1 - \Delta_1 = 11/4 - 9/88 = 233/88 \).
4. (10 points) Let \( f \) be a function whose domain is \( \mathbb{R} \). Suppose that for all \( x, y \in \mathbb{R} \), we have \(|f(x) - f(y)| \leq |x - y|^2\). Prove that \( f \) is constant, i.e. there is a number \( C \in \mathbb{R} \) so that \( f(x) = C \) for all \( x \in \mathbb{R} \).

**Solution.** The solution to this problem consists of two steps. First, we will show that \( f'(c) = 0 \) for all \( c \in \mathbb{R} \). Fix a point \( c \in \mathbb{R} \). Since \( D(f) = \mathbb{R} \), the domain requirement is clearly met. Next, we will show that

\[
\lim_{x \to c} \frac{f(x) - f(c)}{x - c} = 0. \tag{1}
\]

Let \( \varepsilon > 0 \). Select \( \delta = \varepsilon \). Then for all \( x \in \mathbb{R} \) with \( 0 < |x - c| < \delta \), we have

\[
\left| \frac{f(x) - f(c)}{x - c} - 0 \right| = \frac{|f(x) - f(c)|}{|x - c|} \leq \frac{|x - c|^2}{|x - c|} = |x - c| < \varepsilon,
\]

which establishes (1). We conclude that \( f'(x) = 0 \) for all \( x \in \mathbb{R} \).

Next, let \( C = f(0) \); we will show that \( f(x) = C \) for all \( x \in \mathbb{R} \). Fix a point \( x \in \mathbb{R} \) with \( x \neq 0 \). Since \( f \) is differentiable on \( \mathbb{R} \), it is differentiable on the open interval \((0, x)\) (or \((x, 0)\) if \( x < 0 \)). Similarly, since \( f \) is differentiable on \( \mathbb{R} \), it is continuous on \( \mathbb{R} \), and hence it is continuous on \([0, x]\) (or \([x, 0]\) if \( x < 0 \)). Thus by the mean value theorem, there exists a point \( y \in (0, x) \) (or \((x, 0)\) if \( x < 0 \)) so that \( f(x) - f(0) = f'(y)(x - 0) \). But since \( f'(y) = 0 \) for all \( y \in \mathbb{R} \), we have \( f(x) - f(0) = 0 \), i.e. \( f(x) = 0 \). Since this holds for all \( x \in \mathbb{R} \) with \( x \neq 0 \), we conclude that for all \( x \in \mathbb{R} \), \( f(x) = f(0) = C \), as desired.