Math 120 Midterm 2 Practice 1 Solutions

1. Let \((a,b)\) be an interval and let \(c \in (a,b)\). Let \(f\) be a function that is continuous on \((a,b)\)\(\setminus\{c\}\), and suppose \(f\) has an infinite discontinuity at \(c\). Must it be true that \(\lim_{x \to c} |f(x)| = \infty\)? If the answer is yes, prove it. If no, give a counter-example and prove that your counter-example is correct.

Solution. No. Recall that \(f\) has an infinite discontinuity at \(c\) if \(\lim_{x \to c^+} f(x)\) exists (either as a real number or as \(\pm \infty\)); \(\lim_{x \to c^-} f(x)\) exists (either as a real number or as \(\pm \infty\)); and at least one of \(\lim_{x \to c^+} f(x)\) or \(\lim_{x \to c^-} f(x)\) is equal to \(\pm \infty\). Consider the function

\[
f(x) = \begin{cases} 
1/(x-c), & x > c, \\
1, & x \leq c
\end{cases}
\]

Then \(\lim_{x \to c^+} f(x) = \infty\); the domain requirement is clearly met since \(D(f) = \mathbb{R}\). Next, for each \(M \in \mathbb{R}\), if we select \(\delta = 1/M\) then for all \(x \in (c,c+\delta)\) we have \(f(x) = \frac{1}{x-c} > \frac{1}{(c+\delta)-c} = \frac{1}{\delta} = M\).

We also have \(\lim_{x \to c^-} f(x) = 1\); again, the domain requirement is clearly met since \(D(f) = \mathbb{R}\). Next, for each \(\varepsilon > 0\), if we select \(\delta = 1\) then for all \(x \in (c-\delta, c)\), we have \(|f(x) - 1| = |1-1| = 0 < \varepsilon\).

Thus \(f\) has an infinite discontinuity at \(c\). However, note that \(|f(x)| = f(x)|\) for all \(x \in \mathbb{R}\). Thus \(\lim_{x \to c^-} |f(x)| = 1\), which implies that \(\lim_{x \to c} |f(x)|\) cannot be equal to \(\infty\).
2. Let $f$ be a function that is continuous on the interval $[a, b]$ and differentiable on the interval $(a, b)$. Suppose that there exists $M > 0$ so that for each $x \in (a, b)$, we have $|f'(x)| \leq M$. Prove that $|f(b) - f(a)| \leq M|b - a|$.

Solution. By the mean value theorem, there exists a point $x \in (a, b)$ with $f(b) - f(a) = f'(x)(b - a)$. Taking absolute values of both sides, this implies that $|f(b) - f(a)| = |f'(x)||b - a|$. Since $|f'(x)| \leq M$ for all $x \in (a, b)$, we conclude that $|f(b) - f(a)| \leq M|b - a|$.
3. The number \( \frac{1 + \sqrt{5}}{2} \) is called the golden ratio.

a) What are the two integers closest to the golden ratio? You do not have to prove that your answer is correct.

Solution. 1 and 2.

b) Use Newton’s method to approximate the golden ratio. Start with one of the two integers you gave above, and iterate Newton’s method two times (i.e. you should have your starting guess and then two refinements of this guess). You do not need to simplify fractions.

Solution. Consider the polynomial \( f(x) = x^2 - x - 1 \). We can quickly check that if \( x = \frac{1 + \sqrt{5}}{2} \), then

\[
x^2 - x - 1 = \left( \frac{1 + \sqrt{5}}{2} \right)^2 - \left( \frac{1 + \sqrt{5}}{2} \right) - 1 = \frac{1 + 5 + 2\sqrt{5}}{4} - \frac{1 + \sqrt{5}}{2} - 1 = 0.
\]

Let \( x_0 = 2 \). We have

\[
\Delta_0 = \frac{f(x_0)}{f'(x_0)} = \frac{1}{2} = 1/2,
\]

so

\[
x_1 = x_0 - \Delta_0 = 3/2.
\]

Next, we have

\[
\Delta_1 = \frac{f(x_1)}{f'(x_1)} = \frac{-1/4}{3/2} = -1/6,
\]

so

\[
x_2 = x_1 - \Delta_1 = 3/2 + 1/6 = 5/3.
\]
4. Define
\[ f(x) = \begin{cases} 
  x^2, & x \in \mathbb{Q}, \\
-x^2, & x \in \mathbb{R}\setminus\mathbb{Q}
\end{cases} \]

a) Prove that \( f \) is differentiable at 0.

Solution. First, since \( D(f) = \mathbb{R} \), the domain requirement is clearly met. Next, to prove that \( f \) is differentiable at 0, we must show that \( \lim_{x \to 0} \frac{f(x) - f(0)}{x-0} \) exists. Define
\[ g(x) = \begin{cases} 
  x, & x \in \mathbb{Q}, \\
-x, & x \in \mathbb{R}\setminus\mathbb{Q}
\end{cases} \]

Observe that if \( x \neq 0 \), then \( \frac{f(x)}{x} = g(x) \). Thus by the limits are a local property rule, in order to establish that \( \lim_{x \to 0} f(x) \) exists, it suffices to establish that \( \lim_{x \to 0} g(x) \) exists. Next, observe that for all \( x \in \mathbb{R}, -|x| \leq g(x) \leq |x| \). We proved in lecture that \( \lim_{x \to 0} |x| = 0 \), and an identical proof shows that \( \lim_{x \to 0} (-|x|) = 0 \). Thus by the squeeze theorem, \( \lim_{x \to 0} g(x) = 0 \), which completes the proof.

b) Prove that for each \( c \neq 0 \), \( f \) is not differentiable at \( c \).

Solution. Let \( c \neq 0 \). Suppose for contradiction that \( f \) is differentiable at \( c \). Define \( h(x) = 1/x^2 \). Since \( p(x) = x^2 \) is differentiable for all \( x \in \mathbb{R} \), by the reciprocal rule, we have that \( h \) is differentiable at every point \( x \neq 0 \), and in particular \( h \) is differentiable at \( c \). By the product rule for derivatives, this means that the product \( fh \) is differentiable at \( c \), and thus \( fh \) is continuous at \( c \). But
\[ fh(x) = \begin{cases} 
  1, & x \in \mathbb{Q}, x \neq 0 \\
-1, & x \in \mathbb{R}\setminus\mathbb{Q}
\end{cases} \]

and we showed in lecture that this function is not continuous at any point of \( \mathbb{R} \).