Here are some problems prepared in 2018 by Adam Martens to help prepare for the Math 120 final exam. The questions marked with a "*" are more challenging. I would encourage you all to work through as many of them as possible.
1a: Let $\{x_n\}_{n=1}^N \subset \mathbb{R}$ be a (finite) collection of points. Construct a function $f : \mathbb{R} \to \mathbb{R}$ that is continuous at each point x_n for all $n \in \{1, \ldots N\}$ and discontinuous everywhere else.
1b: Using ϵ-δ definition, prove that your function f is continuous at x_1.
***1c:** Using ϵ-δ definition, prove that your function f is continuous at $\{x_n\}_{n=1}^N$.

2: First, a definition. We say that $p \in \mathbb{Z}$ is *prime* if for all $a, b \in \mathbb{Z}$, if $p \mid ab$, then either $p \mid a$ or $p \mid b$ (or possibly both). Here $x \mid y$ means that x divides y, or there exists $a \in \mathbb{Z}$ such that $y = ax$. For this problem, we admit the fact that 3 is prime.
2a: Show $\sqrt[3]{2} \notin \mathbb{Q}$. *Hint: Use the definition of even and odd as in HW 1.*
2b: Show $\sqrt{3} \notin \mathbb{Q}$. *Hint: Modify the proof from HW 1 and use the above definition.*
2c: Show $\sqrt{15} \notin \mathbb{Q}$. *Hint: Your proof from (b) should do the trick.*
2d: Show $\sqrt{2} - \sqrt[3]{2} \notin \mathbb{Q}$. *Hint: This is just some algebra.*

3a: Let $f : (0, \infty) \to \mathbb{R}$ be defined as $f(x) = \sqrt{x}$. Prove (using the definition of the derivative), that $D(f') = d(f) = (0, \infty)$. Of course, in doing so, you should have found a closed form expression for f'.
3b: Use the chain rule and the result of part (a) to find the derivative of $g = \sqrt[4]{x}$ for any $x \in (0, \infty)$.

4: Let $f_a : \mathbb{R} \to \mathbb{R}$ be defined as

$$f_a(x) = \begin{cases} 0 & : \text{ if } x \leq 0 \\ x^a \sin(\frac{1}{x}) & : \text{ if } x > 0 \end{cases}$$

Begin by noticing that f_a is perfectly well defined on $\mathbb{R}$ for all $a \in \mathbb{R}$.
4a: Prove that f_a is continuous at $x = 0$ if and only if $a > 0$. *Hint: Use Squeeze.*
4b: Prove that f_a is differentiable at $x = 0$ if and only if $a > 1$. *Hint: Squeeze again applied to f'.*
4c: Prove that f_a is continuously differentiable at $x = 0$ if and only if $a > 2$.
This function is at the heart of some deep mathematics. For an example, look up *Volterra's function* and read the wiki article. It is fairly technical but may be of interest to some of you.

5: Suppose $f : \mathbb{R} \to \mathbb{R}$ and $a, L \in \mathbb{R}$. For all of the following, carefully state the ϵ, δ, M definition that corresponds to each part.
5a: $\lim_{x \to a} f(x) = L$.
5b: $\lim_{x \to \infty} f(x) = L$.
5c: $\lim_{x \to a} f(x) = \infty$.
5d: $\lim_{x \to a} f(x)$ does not exist.
5e: $\lim_{x \to \infty} f(x) > 0$.

6: Suppose $f : \mathbb{R} \to \mathbb{R}$ is differentiable for all $x \in \mathbb{R}$. That is $D(f') = \mathbb{R}$. Suppose that $\lim_{x \to \infty} f'(x) > 0$. Use the MVT to show that $\lim_{x \to \infty} f(x) = \infty$. (Questions 5 (b) and (e) should tell you exactly what you need to show).

7: Use the ϵ-δ definition to prove the following statements:
7a: $\lim_{x \to 2} x^3 - 3x^2 + x + 4 = 2$.
7b: $\lim_{x \to 1} x^5 - x^4 + x - 1 = 0$.
7c: $\lim_{x \to 0} e^x = 1$.
7d: $\lim_{x \to \pi/2} \sin(x) = 1$.
7e: $\lim_{x \to 2} x^2 - 4 \neq 2$.
7f: $\lim_{x \to 1} x^4 - 3x^3 + 2x^2 - 1 \neq 0$.
7g: $\lim_{x \to 1} e^x \neq 1$.

7h: $\lim_{x\to\pi/2}\cos(x) \neq 1$.

8: Let $S \subset \mathbb{R}$ be a set. Using the quantifiers $\exists$ and $\forall$, write the sentences:
8a: "S has an upper bound."
8b: "S has a least upper bound," and c) "S does not have a least upper bound."

***9:** Notation: $\sup(S)$ is denotes the least upper bound of S, $\inf(S)$ denotes the greatest lower bound of S and for any set $S \subset \mathbb{R}$, let $-S := \{-x : x \in S\}$.
9a: Prove that for any $S \subset \mathbb{R}$, $\sup(S) = -\inf(-S)$.
9b: Let $L = \{x : x \text{ is an upper bound of } S\}$. Prove that, $\sup(S) = \inf(L)$.

10: This question illustrates the importance of the order of quantifiers.
10a: Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that $\forall \epsilon > 0$, $\exists \delta > 0$ such that for all $x, y \in \mathbb{R}$ with $|f(x) - f(y)| < \epsilon \Rightarrow |x - y| < \delta$ but f is not continuous. Prove that your answer is correct.
10a: Let $a \in \mathbb{R}$. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that $\exists \delta > 0$, such that $\forall \epsilon > 0$, and for all $x \in \mathbb{R}$ with $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$. (This is a must stronger condition than simply f being continuous at a).

11: Prove the following by induction. For the rest of this question, n is always a natural number.

11a: $\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$.

11b: $\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$.

11c: $\sum_{i=1}^{n} \frac{1}{i(i+1)} = \frac{n}{n+1}$.

11d: 3 divides $n^3 - n$.

11e: $n! > 2^n$ for all $n \geq 4$.

11f: $\sum_{i=1}^{n}(2i - 1) = n^2$.

11g: $\sum_{i=1}^{n} 2i = n(n + 1)$.

11h: $|\sum_{i=1}^{n} x_n| \leq \sum_{i=1}^{n} |x_n|$ for $x_n \in \mathbb{R}$.

12: One very common mistake that students make is interchanging limits with other operations. The point of this problem is to find some examples of why you cannot just do this freely. Give examples of functions $f, g : \mathbb{R} \to \mathbb{R}$ (defined everywhere by the domain convention) that satisfy each part. After you've come up with an example, recall the appropriate limit rule that applies and prove it yourself using ϵ-δ definition.
12a: $\lim_{x\to a}(f(x) + g(x)) \neq \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$.
12b: $\lim_{x\to a}(f \circ g)(x) \neq f(\lim_{x\to a} g(x))$.
12c: $\lim_{x\to a} f(x)g(x) \neq \lim_{x\to a} f(x) \lim_{x\to a} g(x)$.
12d: $\lim_{x\to a}(f(x)/g(x)) \neq \lim_{x\to a} f(x)/ \lim_{x\to a} g(x)$.

13: For each of the following functions, identify all local/global maxima and minima, inflection points, critical points, asymptotes and whenever the function is increasing/decreasing and concave up or concave down. Sketch the graph (without use of technology)
13a: $\frac{2x^2}{1-x^2}$
13b: $\frac{x^3}{1+x^2}$

13c: $x + \sin(x)$
13d: e^{-x^2} *Note: This is an important function in many areas of math including probability. It's graph is the standard "bell curve".*

14: Find the derivatives of the following functions:
14a: $f(x) = e^{(cos^2(2x+5))}$.
14b: $f(x) = x^{x^x}$.
14c: $f(x) = (x^x)^x$.

15: A function $f : \mathbb{R} \to \mathbb{R}$ is called *even* if $f(x) = f(-x)$ $\forall x \in \mathbb{R}$ and is called *odd* if $f(x) = -f(-x)$ $\forall x \in \mathbb{R}$.
15a: Find the (yes **the** unique) function that is both even and odd.
15b: Prove that the product of two even functions is even.
15c: Prove that the product of two odd functions is even.
15d: Prove that the product of an even and odd function is odd.
15e: Prove that the sum of two even functions is even, and the sum of two odd functions is odd.
15f: Show that any function $f : \mathbb{R} \to \mathbb{R}$ can be written as the sum of an even and odd function.
Note: In addition with the fact that $f(x) = 1$ is even, parts (a),(b),(e) show that the even functions $\mathbb{R} \to \mathbb{R}$ form an algebraic structure called a *Ring*.

16a: Let $f(x) = e^{-x^2}$ (defined on all of $\mathbb{R}$ by domain convention). Find the maximal possible area of a rectangle inscribed under the graph of f. In other words, find the rectangle with one of the sides lying on the x-axis and the other two corners touching the graph of f that has largest area.
16b: A 20 ft ladder is leaning up against a wall. Initial, the base of the ladder is 10 ft away from the way and you are pushing the base toward the wall at a rate of 1/2 ft/second. How fast is the top of the ladder moving up the wall right after we start pushing?
16c: Under the same setup from (b), give a function of the height of the top of the ladder as a function of time. Make sure you specify the domain.

17: Prove that $e^\pi > \pi^e$.

***18:** First, we notice that even though $1/2 \neq 1/4$, we still have $\left(\frac{1}{2}\right)^{\frac{1}{2}} = \left(\frac{1}{4}\right)^{\frac{1}{4}}$. Prove that there are infinitely many pairs of positive real numbers a, b such that $a \neq b$ but $a^a = b^b$. Find all such pairs and prove that you actually found them all.
Hint: consider $f(x) = x^x$.

***19:** Let $f(x) = \sqrt{1 - \sqrt{2 - \sqrt{3 - x}}}$.
19a: Find $D(f)$.
19b: Find $f'(x)$ where it exists.

20a: Find the point on the parabola $y^2 = 2x$ that is closest to the point $(1, 4)$.
20b: Find the point on the curve $y = x^3 - 2x^2 + 1$ that is closest to the point $(2, 2)$.
20c: Find the point on the ellipse $y^2 + 3x^2 = 5$ that is closest to the point $(3, 3)$.

***21:** Let $a \in (0, \infty)$. Suppose you can run a times faster than you can swim. You are standing on the edge of a circular pool of radius R and you want to get to the opposite side. Find your optimal strategy as a function of a.
Hint: Break into cases. Sometimes, you'll have to split time running and swimming.

***22:** Let $f : \mathbb{R} \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} e^{-1/x^2} & : x \neq 0 \\ 0 & : x = 0 \end{cases}$$

Prove (by induction on n) that $f^{(n)}(0) = 0$ for all $n \in \mathbb{N}$. So then the Taylor series of f at $x = 0$ is identically 0. Is this Taylor series a "good" approximation to f?

***23:** Prove that all injective continuous functions defined on all of $\mathbb{R}$ are either strictly increasing or strictly decreasing for all $x \in \mathbb{R}$. Give an example where this fails if the function is not continuous.

***24:** Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that f is differentiable for all $x \neq a$. Prove that if $\lim_{x\to a} f'(x)$ exists, then f is differentiable at $x = a$.

25: For each of the following, find the least upper bound of the set S. Remember, you need to show that if x is the LUB of S, then (1) x is an upper bound and (2), for any $a < x$, a is not an upper bound.
25a: $S = \{1 - 1/n^2 : n \in \mathbb{N}\}$.
25b: $S = \{1/a - 1/b : a, b \in \mathbb{N}\}$.
25c: $S = \{1/n + 1/m : n, m \in \mathbb{N}\}$.

26: Compute the following:
26a: $\lim_{x\to\infty} x(\log(x + 5) - \log(x))$.
26b: $\lim_{x\to 0} \frac{e^x - 3 + 2\cos(x) + x^2}{x^4}$.
26c: $\lim_{x\to 0} x^{\sin(x)}$.
26d: $\lim_{x\to 0} \frac{e^{2x} + x^5 - \cos(x^2)}{x^4 - 1}$.

27: Suppose $f(1) = f'(1) = 1$ and $x \leq f''(x) \leq 2 - x^2$ on the interval $[1/2, 1]$. Find the smallest interval you can to be sure contains $f(1/2)$.