Math 120 Homework 9

- Due Friday November 22th at 10 am. Homework should be submitted using Canvas.
- **Collaboration Policy**: You are welcome (and encouraged) to work on the homework in groups. However, each student must write up the homework on their own, and must use their own wording (i.e. don't jusy copy the solutions from your friend). If you do collaborate with others, please list the name of your collaborators at the top of the homework.
- You are encouraged (though not required) to type up your solutions. If you choose to do this, I strongly recommend that you use the typesetting software LaTeX. LaTeX is used by the entire mathematics community, and if you intend to go into math, youll need to learn it sooner or later. "The Not So Short Introduction to LaTeX" is a good place to start. This guide can be found at http://tug.ctan.org/info/lshort/english/lshort.pdf . You can also download the .tex source file for this homework and take a look at that.
- Each homework problem should be correct as stated. Occasionally, however, I might screw something up and give you an impossible homework problem. If you believe a problem is incorrect, please email me. If you are right, the first person to point out an error will get +1 on that homework, and I will post an updated version.

1 a. Prove that $x^3 + x^2 + 10 = O(1)$ as $x \to 0$.

b. Prove that $x^2 + x = O(x)$ as $x \to 0$.

c. Prove that $x^2 + x = O(1)$ as $x \to 0$.

d. Prove that the statement "$1 = O(x^2)$ as $x \to 0$" is false.

2 a. Prove that $\cos(x) = O(1)$ as $x \to \infty$

b. Prove that $x^2 + \sin(3x) + 1 = O(x^2)$ as $x \to \infty$.

c. Prove that $x^2 + \sin(3x) + 1 = O(x^3)$ as $x \to \infty$.

3. Let f be a function whose domain is $\mathbb{R}$. Let $n \in \mathbb{N}$. Suppose that f is n times differentiable on $\mathbb{R}$ and that $f^{(n)}(x) = 0$ for every $x \in \mathbb{R}$. Prove that f is a polynomial of degree at most $n - 1$.

For the next few problems, we will use the following definition. Let f be a function that is n times differentiable on $[a, b]$ and whose n-th derivative is continuous on $[a, b]$. Let $c \in [a, b]$. We define the "degree n Taylor expansion of $f(x)$ around c" to be the function

$$h(x) = f(c) + \sum_{k=1}^{n} \frac{f^{(k)}(c)}{k!}(x - c)^k.$$

4. Compute the degree 3 Taylor expansion of $f(x) = x^5 + 2x^4 + 3x^3 + 4x^2 + 5x + 1$ around the point $c = 0$. You don't need to prove that your answer is correct, but do show your work.

5. Compute the degree 3 Taylor expansion of $f(x) = x^5 + 2x^4 + 3x^3 + 4x^2 + 5x + 1$ around the point $c = 2$. You don't need to prove that your answer is correct, but do show your work.

6. a. Prove that for each even natural number $n \in \mathbb{N}$, $n! \geq (n/2)^{n/2}$.

b. Prove that for each $x \in \mathbb{R}$, and each $\varepsilon > 0$, there exists a number $n \in \mathbb{N}$ so that $\left|\frac{x^n}{n!}\right| < \varepsilon$.
c. Prove that for each $x > 0$ and each $\varepsilon > 0$, there exists a number $n \in \mathbb{N}$ so that

$$\left| \cos(y) - \left(1 + \sum_{\substack{k=1 \\ k \text{ even}}}^{n} \frac{(-1)^{k/2}}{k!} y^k\right) \right| < \varepsilon.$$

for all $y \in [-x, x]$.