Math 120 Homework 8 Solutions

In this homework, you will construct a rather strange function.

1. Prove that if $P(x)$ is a polynomial, then
   \[ \lim_{x \to 0^+} P(1/x) e^{-\frac{1}{x}} = 0 \]

   **Solution.** Recall from HW 4 #2b that if $g(x) = 1/x$ and if $\lim_{x \to \infty} f \circ g(x) = L$, then $\lim_{x \to 0^+} f(x) = L$. Let $f(x) = P(\frac{1}{x}) e^{-\frac{1}{x}}$. Then if $x \neq 0$, $f \circ g(x) = P(x)e^{-x} = P(x)/e^x$. We proved in lecture that $\lim_{x \to \infty} f \circ g(x) = 0$, and thus $\lim_{x \to 0^+} f(x) = 0$.

2. Let $P(x) = a_n x^n + \ldots + a_0$ be a polynomial and let $f(x) = P(1/x)$. Prove that there exists a polynomial $Q(x)$ so that for all $x \neq 0$,
   \[ f'(x) = Q(1/x). \]

   **Solution.** We have
   \[ f(x) = a_n x^{-n} + a_{n-1} x^{-(n-1)} + \ldots + a_1 x^{-1} + a_0. \]
   We proved in class (using the quotient rule) that if $f(x) = x^{-k}$, then $f'(x) = -kx^{-k-1}$ for all $x \neq 0$. Thus using the sum rule for derivatives, we have that for all $x \neq 0$,
   \[ f'(x) = a_n(-n)x^{-n-1} + a_{n-1}(-n-1)x^{-n} + \ldots + a_1(-1)x^{-2}. \]

   Thus if we define
   \[ Q(x) = -na_n x^{n+1} - (n-1)a_{n-1} x^n - (n-2)a_{n-2} x^{n-1} - \ldots - a_1 x^{-2}, \]
   then $f'(x) = Q(1/x)$ for all $x \neq 0$.

3. $P(x) = a_n x^n + \ldots + a_0$ be a polynomial and let $g(x) = P(1/x)e^{-1/x}$. Prove that there exists a polynomial $R(x)$ so that for all $x \neq 0$,
   \[ g'(x) = R(1/x)e^{-1/x}. \]

   **Solution.** Let $P(x) = a_n x^n + \ldots + a_0$ and let $f(x) = P(1/x)$. Note that both $f(x)$ and $e^{-1/x}$ are differentiable for $x \neq 0$. Using the product rule, we have that
   \[ g'(x) = f'(x)e^{-1/x} + f(x)(e^{-1/x})' = (f'(x) + f(x) \cdot (-1/x^2))e^{-1/x} \]

   By problem 2, we have that there exits a polynomial $Q(x) = b_m x^m + \ldots + b_1$ so that $f'(x) = Q(1/x)$ for all $x \neq 0$. Define
   \[ R(x) = Q(x) - x^2 P(x) = (b_m x^m + \ldots + b_1) - (a_n x^{n+2} + a_{n-1} x^{n+1} + \ldots + a_0 x^2). \]
   This is clearly a polynomial, and $R(1/x) = f'(x) - (1/x^2)f(x)$ for all $x \neq 0$, so
   \[ g'(x) = R(1/x)e^{-1/x} \]
for all $x \neq 0$.

4. Prove by induction that for each integer $n \geq 1$, there exists a polynomial $R_n(x)$ so that if $f(x) = e^{-1/x}$, then

$$f^{(n)}(x) = R_n(1/x)e^{-1/x}.$$  

for all $x \neq 0$.

Solution.

First we will do the base case. If $n = 1$, then $f^{(1)}(x) = (-1/x^2)e^{-1/x}$, so the result is true with $R_1(x) = -x^2$.

Next we will do the induction step. Suppose the result has been proved for some integer $n \geq 1$. Then for all $x \neq 0$,

$$f^{(n+1)}(x) = (f^{(n)}(x))' = (R_n(1/x)e^{-1/x})' = R_{n+1}(1/x)e^{-1/x}.$$  

For the last inequality we used Problem 3 (with $P(x) = R_n(x)$), and we define $R_{n+1}$ to be the output polynomial $R$ from problem 3. This completes the induction step and thus completes the proof.

5. Define

$$g(x) = \begin{cases} 0, & x \leq 0, \\ e^{-1/x}, & x > 0. \end{cases}$$

a. Prove that for every number $n$, $g$ is $n$-times differentiable on $\mathbb{R}$ (i.e. $g^{(n)}(x)$ exists for every $x \in \mathbb{R}$).

b. Prove that $g^{(n)}(0) = 0$ for every non-negative integer $n = 1, 2, \ldots$.

Solution.

a. Since $g(x) = 0$ if $x < 0$, by the limits are a local property rule we have that $g^{(n)}(x) = 0$ if $x < 0$. Similarly, since $g(x) = e^{-1/x}$ if $x > 0$, by the limits are a local property rule and Problem 4, we have that $g^{(n)}(x) = R_n(1/x)e^{-1/x}$ if $x > 0$, where $R_n$ is the polynomial from problem 4. In particular, we know that for each integer $n \geq 1$, $g$ is $n$ times differentiable at $x$ for all $x \neq 0$. In part b below, we will show that $g$ is $n$ times differentiable at 0.

b. We will prove by induction on $n$ that $g^{(n)}(0) = 0$ for all $n$, so in particular, $g$ is $n$ times differentiable at 0. We will begin with $n = 0$. Since $g(x)$ is defined for all $x \in \mathbb{R}$, $g(x)$ is 0-times differentiable at 0, and $g^{(0)}(0) = g(0) = 0$. Now suppose we have shown that $g^{(n)}(0) = 0$. In order to prove that $g^{(n+1)}(0) = 0$, it suffices to prove that

$$\lim_{x \to 0^-} \frac{g^{(n)}(x) - g^{(n)}(0)}{x} = \lim_{x \to 0^+} \frac{g^{(n)}(x) - g^{(n)}(0)}{x} = 0.$$  

Since $g^{(n)}(0) = 0$ and $g^{(n)}(x) = 0$ for all $x < 0$, we have $\lim_{x \to 0^-} \frac{g^{(n)}(x) - g^{(n)}(0)}{x} = 0$. Since $g^{(n)}(x) = R_n(1/x)e^{-1/x}$ for all $x > 0$ (here $R_n$ is the polynomial from Problem 4), we have

$$\lim_{x \to 0^+} \frac{g^{(n)}(x) - g^{(n)}(0)}{x} = \lim_{x \to 0^+} \frac{1}{x}R_n(1/x)e^{-1/x} = 0,$$

where for the last equality we used Problem 1. This completes the induction step and finishes the proof.
6. Let \( f \) and \( g \) be functions that are differentiable on \((0, \infty)\). Suppose that \( f \) and \( g \) satisfy the functional equations

\[
\begin{align*}
  f(xy) &= f(x) + f(y) \quad \text{for all } x, y \in (0, \infty), \\
  g(xy) &= g(x) + g(y) \quad \text{for all } x, y \in (0, \infty),
\end{align*}
\]

and that \( f'(1) = g'(1) \).

a. Prove that \( f'(x) = g'(x) \) for all \( x \in (0, \infty) \).

b. Prove that \( f(x) = g(x) \) for all \( x \in (0, \infty) \).

**Remark** This problem establishes that there is one function satisfying \( f(xy) = f(x) + f(y) \) and \( f'(1) = 1 \). Thus Euler’s number \( e \) is well defined.

**Solution**

a. In lecture, we proved that if \( f \) is differentiable for all \( x \in (0, \infty) \) and satisfies \( f(xy) = f(x) + f(y) \) for all \( x, y \in (0, \infty) \), then for each \( x \in (0, \infty) \) we have \( f'(x) = f'(1)/x \). The same reasoning shows that \( g'(x) = g'(1)/x \). Thus if \( f'(1) = g'(1) \), we conclude that \( f'(x) = g'(x) \) for all \( x \in (0, \infty) \).

b. Since \( f'(x) = g'(x) \) for all \( x \in (0, \infty) \), we have that \( (f - g)'(x) = 0 \) for all \( x \in (0, \infty) \). In lecture, we proved that if \( f \) is differentiable for all \( x \in (0, \infty) \) and satisfies \( f(xy) = f(x) + f(y) \) for all \( x, y \in (0, \infty) \), then \( f(1) = 0 \). Similarly, \( g(1) = 0 \). This means that \( (f - g)(1) = 0 \).

Now, let \( x \in (0, \infty) \) with \( x \neq 1 \). By the mean value theorem applied to \( h = f - g \), there exists a point \( c \) in the interval between 1 and \( x \) so that \( (f - g)(x) - (f - g)(1) = (f - g)'(c)(x - 1) = 0(x - 1) = 0 \). We conclude that \( (f - g)(x) = (f - g)(1) = 0 \), i.e. \( (f - g)(x) = 0 \) for all \( x \in (0, \infty) \). Hence \( f(x) = g(x) \) for all \( x \in (0, \infty) \).