Math 120 Homework 8

- Due Friday November 15th at 10 am. Homework should be submitted using Canvas.
- **Collaboration Policy**: You are welcome (and encouraged) to work on the homework in groups. However, each student must write up the homework on their own, and must use their own wording (i.e. don't jusy copy the solutions from your friend). If you do collaborate with others, please list the name of your collaborators at the top of the homework.
- You are encouraged (though not required) to type up your solutions. If you choose to do this, I strongly recommend that you use the typesetting software LaTeX. LaTeX is used by the entire mathematics community, and if you intend to go into math, youll need to learn it sooner or later. "The Not So Short Introduction to LaTeX" is a good place to start. This guide can be found at http://tug.ctan.org/info/lshort/english/lshort.pdf . You can also download the .tex source file for this homework and take a look at that.
- Each homework problem should be correct as stated. Occasionally, however, I might screw something up and give you an impossible homework problem. If you believe a problem is incorrect, please email me. If you are right, the first person to point out an error will get +1 on that homework, and I will post an updated version.

1. Prove that if $P(x)$ is a polynomial, then

$$\lim_{x\to 0^+} P(1/x)e^{-\frac{1}{x}} = 0.$$

2. Let $P(x) = a_n x^n + \ldots + a_0$ be a polynomial and let $f(x) = P(1/x)$. Prove that there exists a polynomial $Q(x)$ so that for all $x \neq 0$,

$$f'(x) = Q(1/x).$$

3. $P(x) = a_n x^n + \ldots + a_0$ be a polynomial and let $g(x) = P(1/x)e^{-1/x}$. Prove that there exists a polynomial $R(x)$ so that for all $x \neq 0$,

$$g'(x) = R(1/x)e^{-1/x}.$$

4. Prove by induction that for each integer $n \geq 1$, there exists a polynomial $R_n(x)$ so that if $f(x) = e^{-1/x}$, then

$$f^{(n)}(x) = R_n(1/x)e^{-1/x}.$$

for all $x \neq 0$.

5. Define

$$g(x) = \begin{cases} 0, & x \leq 0, \\ e^{-1/x} & x > 0 \end{cases}$$

a. Prove that for every number n, g is n-times differentiable on $\mathbb{R}$ (i.e. $g^{(n)}(x)$ exists for every $x \in \mathbb{R}$).

b. Prove that $g^{(n)}(0) = 0$ for every non-negative integer $n = 1, 2, \ldots$.

6. Let f and g be functions that are differentiable on $(0, \infty)$. Suppose that f and g satisfy the functional equations

$$f(xy) = f(x) + f(y) \quad \text{for all } x, y \in (0, \infty),$$
$$g(xy) = g(x) + g(y) \quad \text{for all } x, y \in (0, \infty),$$

and that $f'(1) = g'(1)$.

a. Prove that $f'(x) = g'(x)$ for all $x \in (0, \infty)$.

b. Prove that $f(x) = g(x)$ for all $x \in (0, \infty)$.

Remark This problem establishes that there is *one* function satisfying $f(xy) = f(x) + f(y)$ and $f'(1) = 1$. Thus Euler's number e is well defined.