Math 120 Homework 7 Solutions

1. a. Let \( n \) and \( k \) be integers with \( 1 \leq k < n \) and let \( f(x) = x^n \). Prove that \( f^{(k)}(x) = n(n-1)(n-2) \cdots (n-k+1)x^{n-k} \). Hint: induction

b. Prove that if \( f(x) = x^n \), then \( f^{(n)}(x) = n(n-1)(n-2) \cdots 1 \).

Remark: The expression \( n(n-1)(n-2) \cdots 1 \) is also called \( n! \), which is pronounced “\( n \) factorial.”

c. Prove that if \( k > n \) and if \( f(x) = x^n \), then \( f^{(k)}(x) = 0 \).

Solution a. Fix a number \( n \geq 2 \); we will prove the result by induction on \( k \). When \( k = 1 \), we proved in lecture that \( f^{(1)}(x) = nx^{n-1} \) whenever \( n \geq 2 \). Now suppose we have already proved that \( f^{(k)}(x) = n(n-1)(n-2) \cdots (n-k+1)x^{n-k} \). If \( k \geq n-2 \) then there is nothing more to prove. If \( k < n-2 \), then

\[
\begin{align*}
f^{(k+1)}(x) &= (f^{(k)}(x))' \\
&= (n(n-1)(n-2) \cdots (n-k+1)x^{n-k})' \\
&= n(n-1)(n-2) \cdots (n-k+1)(n-k)x^{n-k-1},
\end{align*}
\]

which completes the induction step.

b. From part a, we know that \( f^{(n-1)}(x) = n(n-1)(n-2) \cdots (2)(1)x \). Differentiating one more time, we conclude that \( f^{(n)}(x) = n(n-1)(n-2) \cdots (2)(1) = n! \).

c. From part b, we know that \( f^{(n)}(x) = n! \), which is a constant function. We will now prove by induction that for all \( k > n \), \( f^{(k)}(x) = 0 \). For the base case, let \( k = n + 1 \). Then \( f^{(k)}(x) = f^{(n+1)}(x) = 0 \), since the derivative of a constant function is 0. For the induction step, suppose that \( f^{(k)}(x) = 0 \). Then \( f^{(k+1)}(x) = 0 \), since the derivative of a constant function is 0. We conclude that \( f^{(k)}(x) = 0 \) for all \( k > n \).

2. Use Newton’s method with an initial guess \( x_1 = 1 \) to approximate the (positive) fourth root of 2, and show your computations. It’s enough to iterate 3 times (i.e. compute \( x_2, x_3, x_4 \)), and feel free to use a calculator.

Solution Let \( f(x) = x^4 - 2 \). We wish to find a positive number \( x \) with \( f(x) = 0 \). Using Newton’s method, we have

\[
x_1 = 1; \ f(x_1) = -1; \ f'(x_1) = 4; \ \Delta_1 = f(x_1)/f'(x_1) = -1/4; \ x_2 = x_1 - \Delta_1 = 1 - (-1/4) = 5/4.
\]

\[
x_2 = 5/4; \ f(x_2) = 113/256; \ f'(x_2) = 125/16; \ \Delta_2 = f(x_2)/f'(x_2) = 113/2000; \ x_3 = x_2 - \Delta_2 = 5/4 - 113/2000 = 2387/2000.
\]

\[
x_3 = 2387/2000; \ f(x_3) \sim 0.0290357; \ f'(x_3) \sim 6.80029; \ \Delta_3 = f(x_3)/f'(x_3) \sim 0.00426977; \ x_4 = x_3 - \Delta_3 \sim 1.18923.
\]

So \( x_4 \sim 1.18923 \).

3. Let \( f \) be a function that is continuous at every point \( x \in \mathbb{R} \) and is \( 2\pi \) periodic: This means that for all \( x \in \mathbb{R} \), \( f(x) = f(x + 2\pi) \). Let \( g(x) = f(x + \pi) - f(x) \).

a. Show that there is a point \( a \in [0, 2\pi) \) such that \( g(a) = 0 \).

b. Deduce that on the equator of the earth, there are always two points diametrically opposed with the same temperature.
Solution

a. Observe that \( g(x) \) is continuous, since it is the difference of two continuous functions. \( g(0) = f(0) - f(\pi) \) and \( g(\pi) = f(\pi) - f(2\pi) = f(\pi) - f(0) = -g(0) \). If \( g(0) = 0 \) then we are done. If not, then by the intermediate value theorem there exists \( x \in [0, \pi] \) so that \( g(x) = 0 \).

b. For each \( x \in \mathbb{R} \), let \( f(x) \) be the temperature of the earth at \( x \) radians from the prime meridian. Then \( f(x) = f(x + 2\pi) \) for all \( x \in \mathbb{R} \), so \( f \) is \( 2\pi \) periodic. By part a, there exists \( x \in \mathbb{R} \) so that \( g(x) = 0 \). This means that the diametrically opposed points \( x \) and \( x + \pi \) have the same temperature.

4. Let \( b \in \mathbb{R} \) and define \( f(x) = x^3 - 3x + b \). Prove that there is at most one point \( x \in [-1, 1] \) with \( f(x) = 0 \). Note: you don’t get to choose \( b \); you must prove the result for any \( b \in \mathbb{R} \). Hint: Rolle’s theorem.

Solution. We will do a proof by contradiction. Suppose that there exist two points, \( x_1, x_2 \in [-1, 1] \) with \( f(x_1) = 0 \) and \( f(x_2) = 0 \). Assume that \( x_1 < x_2 \) (if not reverse the roles of \( x_1 \) and \( x_2 \)). If we restrict \( f \) to the interval \([x_1, x_2]\), then \( f \) is differentiable on \([x_1, x_2]\) and \( f(x_1) = f(x_2) = 0 \), so by Rolle’s theorem, there exists a point \( c \in (x_1, x_2) \subset (-1, 1) \) with \( f'(c) = 0 \). But \( f'(c) = 3c^2 - 3 \). Since \( c^2 < 1 \) for \( x \in (-1, 1) \), \( 3c^2 - 3 < 0 \) for all \( x \in (-1, 1) \), so there cannot exist a point \( c \in (-1, 1) \) with \( f'(c) = 0 \). This is a contradiction; we conclude that there exists at most one point \( x \in [-1, 1] \) with \( f(x) = 0 \).

5. Let \( f \) be a function whose domain is \( \mathbb{R} \) and that satisfies the functional equation

\[
f(x + y) = f(x) + f(y) \quad \text{for all } x, y \in \mathbb{R}.
\]

This is called Cauchy’s functional equation.

a. Prove by induction that for all \( n \in \mathbb{N} \), \( f(n) = nf(1) \).

b. Prove that for all \( n \in \mathbb{Z} \), we have \( f(n) = nf(1) \).

c. Prove that for all \( q \in \mathbb{Q} \), \( f(q) = qf(1) \).

d. Suppose that \( f \) is continuous at every point \( x \in \mathbb{R} \). Prove that for all \( x \in \mathbb{R} \), \( f(x) = xf(1) \).

e. Bonus: If we do not require that \( f \) is continuous, must it still be the case that \( f(x) = xf(1) \)? Hint: read about discontinuous additive functions.

Solution a. We will prove the result by induction on \( n \). The base case \( n = 1 \) is immediate, since \( f(1) = 1 \cdot f(1) \). Now suppose that we have already proved that \( f(n) = nf(1) \). We have \( f(n+1) = f(n) + f(1) = nf(1) + f(1) = (n+1)f(1) \).

b. First, observe that \( f(0) = f(0+0) = f(0) + f(0) = 2f(0) \), which implies \( f(0) = 0 \). Next, \( n \in \mathbb{Z} \) with \( n \neq 0 \). If \( n > 0 \) then the result was already proved in part a. If \( n < 0 \), then \( 0 = f(0) = f(n + (−n)) = f(n) + f(−n) = f(n) + (−n)f(1) \). We conclude that \( f(n) = nf(1) \).

c. First, we will prove by induction that if \( x \in \mathbb{Q} \) the \( f(nx) = nf(x) \). The base case \( n = 1 \) is immediate. Now suppose \( f(nx) = nf(x) \). Then \( f((n+1)x) = f(nx + x) = f(nx) + f(x) = nf(x) + f(x) = (n+1)f(x) \).

Next, let \( q = a/b \) be rational. We have \( bf(q) = f(bq) = f(a) = af(1) \), and thus \( f(q) = (a/b)f(1) = qf(1) \).

d. Observe that the function \( g(x) = xf(1) \) is continuous; the function \( f(x) \) is continuous, and \( f(x) = g(x) \) for all \( x \in \mathbb{Q} \). Thus by HW4 #4, we conclude that \( f(x) = g(x) \) for all \( x \in \mathbb{R} \).

e. No; there are examples of functions \( f \) satisfying \( f(x+y) = f(x) + f(y) \) that are not of the form \( f(x) = xf(1) \). Read about discontinuous additive functions to learn more.