Math 120 Homework 7

- Due Friday November 1st at 10 am. Homework should be submitted using Canvas.
- **Collaboration Policy**: You are welcome (and encouraged) to work on the homework in groups. However, each student must write up the homework on their own, and must use their own wording (i.e. don't jusy copy the solutions from your friend). If you do collaborate with others, please list the name of your collaborators at the top of the homework.
- You are encouraged (though not required) to type up your solutions. If you choose to do this, I strongly recommend that you use the typesetting software LaTeX. LaTeX is used by the entire mathematics community, and if you intend to go into math, youll need to learn it sooner or later. "The Not So Short Introduction to LaTeX" is a good place to start. This guide can be found at http://tug.ctan.org/info/lshort/english/lshort.pdf . You can also download the .tex source file for this homework and take a look at that.
- Each homework problem should be correct as stated. Occasionally, however, I might screw something up and give you an impossible homework problem. If you believe a problem is incorrect, please email me. If you are right, the first person to point out an error will get +1 on that homework, and I will post an updated version.

1 a. Let n and k be integers with $1 \leq k < n$ and let $f(x) = x^n$. Prove that $f^{(k)}(x) = n(n-1)(n-2)\cdots(n-k+1)x^{n-k}$. Hint: induction

b. Prove that if $f(x) = x^n$, then $f^{(n)}(x) = n(n-1)(n-2)\cdots 1$.
Remark: The expression $n(n-1)(n-2)\cdots 1$ is also called $n!$, which is pronounced "n factorial."

c. Prove that if $k > n$ and if $f(x) = x^n$, then $f^{(k)}(x) = 0$.

2. Use Newton's method with an initial guess $x_1 = 1$ to approximate the (positive) fourth root of 2, and show your computations. It's enough to iterate 3 times (i.e. compute x_2, x_3, x_4), and feel free to use a calculator.

3. Let f be a function that is continuous at every point $x \in \mathbb{R}$ and is 2π periodic: This means that for all $x \in \mathbb{R}$, $f(x) = f(x + 2\pi)$. Let $g(x) = f(x + \pi) - f(x)$.

a. Show that there is a point $a \in [0, 2\pi)$ such that $g(a) = 0$.
b. Deduce that on the equator of the earth, there are always two points diametrically opposed with the same temperature.

4. Let $b \in \mathbb{R}$ and define $f(x) = x^3 - 3x + b$. Prove that there is at most one point $x \in [-1, 1]$ with $f(x) = 0$. Note: you don't get to choose b; you must prove the result for any $b \in \mathbb{R}$. Hint: Rolle's theorem.

5.Let f be a function whose domain is $\mathbb{R}$ and that satisfies the functional equation

$$f(x + y) = f(x) + f(y) \quad \text{for all } x, y \in \mathbb{R}.$$

This is called Cauchy's functional equation.

a. Prove by induction that for all $n \in \mathbb{N}$, $f(n) = nf(1)$.

b. Prove that for all $n \in \mathbb{Z}$, we have $f(n) = nf(1)$.

c. Prove that for all $q \in \mathbb{Q}$, $f(q) = qf(1)$.

d. Suppose that f is continuous at every point $x \in \mathbb{R}$. Prove that for all $x \in \mathbb{R}$, $f(x) = xf(1)$.

e. Bonus: If we do not require that f is continuous, must it still be the case that $f(x) = xf(1)$? Hint: read about discontinuous additive functions.