Math 120 Homework 6

- Due Friday October 25th at 10 am. Homework should be submitted using Canvas.
- **Collaboration Policy**: You are welcome (and encouraged) to work on the homework in groups. However, each student must write up the homework on their own, and must use their own wording (i.e. don't jusy copy the solutions from your friend). If you do collaborate with others, please list the name of your collaborators at the top of the homework.
- You are encouraged (though not required) to type up your solutions. If you choose to do this, I strongly recommend that you use the typesetting software LaTeX. LaTeX is used by the entire mathematics community, and if you intend to go into math, youll need to learn it sooner or later. "The Not So Short Introduction to LaTeX" is a good place to start. This guide can be found at http://tug.ctan.org/info/lshort/english/lshort.pdf . You can also download the .tex source file for this homework and take a look at that.
- Each homework problem should be correct as stated. Occasionally, however, I might screw something up and give you an impossible homework problem. If you believe a problem is incorrect, please email me. If you are right, the first person to point out an error will get +1 on that homework, and I will post an updated version.

In lecture, we will prove that if $f(x) = \sin(x)$, then $f'(x) = \cos(x)$, and if $f(x) = \cos(x)$, then $f'(x) = -\sin(x)$. For the next problem you may use these facts freely.

1. Let

$$f(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

a. Prove that f is differentiable at every point $x \in \mathbb{R}$ (and thus $D(f') = \mathbb{R}$). Hint: consider the case where $x = 0$ and $x \neq 0$ separately.

b. Prove that f' is continuous at every point $x \neq 0$, and is discontinuous at $x = 0$.

c. Prove that if g and h are functions and $x \in \mathbb{R}$ so that g is continuous at x and h is discontinuous at x, then $g + h$ is discontinuous at x.

d. Give an example of a function g that is differentiable at every point $x \in \mathbb{R}$, but whose derivative g' is discontinuous on the set $\{1, 2, 3\} \subset \mathbb{R}$.

2. Let f be a function whose domain is $[0, 1]$ and suppose that f is continuous on $[0, 1]$ and $0 \leq f(x) \leq 1$ for each $x \in [0, 1]$. Prove that there exists a point $c \in [0, 1]$ with $f(c) = c$.

3. Let f be a function and suppose that (a, b) is an interval contained in the domain of f, i.e. $(a, b) \subset D(f)$. Let $c \in (a, b)$ and suppose that f is differentiable at c and that $f'(c) > 0$. Prove that there exists a number $t > 0$ so that: for all $x \in (c, c + t)$, we have $f(x) > f(c)$, and for all $x \in (c - t, c)$, we have $f(x) < f(c)$.

4. Let $f(x) = 2x^5 - 3x^3 + 2x - 2$. In general, there is no closed form expression for the roots of a polynomial of degree 5 or higher. However, note that $f(1) = -1 < 0$ and $f(2) = 42 > 0$. Thus by the intermediate value theorem, there must exist some point $x \in (1, 2)$ with $f(x) = 0$. Here is an algorithm for computing an approximation for x:

- At step zero, let $a_0 = 1$ and $b_0 = 2$. Let $c_0 = (a_0 + b_0)/2 = 3/2$.
- Evaluate $f(c_0)$. If it is positive, let $a_1 = a_0$ and let $b_1 = c_0$. If it is negative, let $a_1 = c_0$ and let $b_1 = b_0$. Note that in either case, we have $f(a_1) < 0$ and $f(b_1) > 0$. Thus there is a point $x \in (a_1, b_1)$ with $f(x) = 0$.
- Evaluate $f(c_1)$. If it is positive, let $a_2 = a_1$ and let $b_2 = c_1$. If it is negative, let $a_2 = c_1$ and let $b_2 = b_1$. Note that in either case, we have $f(a_2) < 0$ and $f(b_2) > 0$. Thus there is a point $x \in (a_2, b_2)$ with $f(x) = 0$.
- Keep repeating this process. At each stage, we know that there is a point $x \in (a_n, b_n)$ with $f(x) = 0$. Furthermore, $|b_n - a_n| \leq 1/2^n$.

Use this algorithm to compute a point $x \in \mathbb{R}$ with $f(x) = 0$ to three significant digits (i.e. your answer should be of the form $1.ab$ (e.g. 1.23)–make sure that both digits a and b are correct!).