Math 120 Homework 5 Solutions

1 a. Let \( P(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0 \) be a polynomial. Suppose that \( a_n > 0 \) and \( a_0 < 0 \). Prove that there exists a point \( x \in \mathbb{R} \) with \( P(x) = 0 \).

Solution. a. We proved in lecture that polynomials are continuous. Observe that \( f(0) = a_0 < 0 \). We will find a point \( b \in \mathbb{R} \) with \( f(b) > 0 \). Once we have done so, Bolzano’s theorem guarantees that there exists a point \( x \in (a, b) \) with \( f(x) = 0 \). In particular, there exists a point \( x \in \mathbb{R} \) with \( f(x) = 0 \).

Let \( B = |a_{n-1}| + |a_{n-2}| + \ldots + |a_0| \). Then for each \( x \in \mathbb{R} \) with \( x \geq 1 \), we have

\[
|a_{n-1} x^{n-1} + \ldots + a_0| \leq |a_{n-1} x^{n-1} + |a_{n-2}| x^{n-2} + \ldots + |a_0| |
\leq |a_{n-1} x^{n-1} + |a_{n-2}| x^{n-2} + \ldots + |a_0||x^{n-1}|
= B x^{n-1}.
\]

Next, observe that for every two real numbers \( y, z \in \mathbb{R} \), we have \( y + z \geq y - |z| \). Setting \( y = a_n x^n \) and \( z = a_{n-1} x^{n-1} + \ldots + a_0 \), we have that if \( x > 1 \) then

\[
f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0
\geq a_n x^n - |a_{n-1} x^{n-1} + \ldots + a_0|
\geq a_n x^n - B x^{n-1}
= (a_n x - B) x^{n-1}.
\]

Thus if \( x > \text{max}(1, a_n/B) \), we have that \( f(x) > 0 \). Observe as well that if \( x < -1 \) then

\[
f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0
\leq a_n x^n + |a_{n-1} x^{n-1} + \ldots + a_0|
\leq a_n x^n - B |x|^{n-1}.
\]

Let \( b = \text{max}(1, a_n/B) \). Then \( f(0) < 0 \) and \( f(b) > 0 \), so by Bolzano’s theorem, there exists a point \( x \in (0, b) \) with \( f(x) = 0 \).

b. Let \( P(x) \) be a polynomial of odd (and positive) degree. Prove that for every \( y \in \mathbb{R} \), there exists a point \( x \in \mathbb{R} \) with \( P(x) = y \).

Solution. Let \( y \in \mathbb{R} \). Since \( f \) is continuous, by the intermediate value theorem it suffices to find points \( x_1, x_2 \in \mathbb{R} \) so that \( f(x_1) < y < f(x_2) \). In (1) we proved that

\[
f(x) \geq a_n x^n - B x^{n-1}
\]
whenever \( x \geq 1 \), and thus \( f(x) \geq (a_n/2)x^n \geq (a_n/2)x \) whenever \( x \geq \text{max}(1, 2B/a_n) \). In (2) we showed that

\[
f(x) \leq a_n x^n + B |x|^{n-1}
\]
whenever \( x \leq -1 \). In particular, since \( n \) is odd and \( a_n > 0 \), this implies \( f(x) \leq (a_n/2)x^n \leq (a_n/2)x \) whenever \( x \leq \text{min}(-1, -2B/a_n) \). Thus if we select \( x_1 = -2B/a_n - 2|y|/a_n - 1 \) and...
$x_2 = 2B/a_n + 2|y|/a_n + 1$, then $x_1 \leq \min(-2B/a_n, -1)$, so

$$f(x_1) \leq (a_n/2)(x_1)$$
$$= (a_n/2)(-2B/a_n - 2|y|/a_n - 1)$$
$$< (a_n/2)(-2|y|/a_n)$$
$$= -|y|$$
$$\leq y.$$

Similarly, $x_2 \geq \max(2B/a_n, 1)$, so

$$f(x_2) \geq (a_n/2)(x_1)$$
$$= (a_n/2)(2B/a_n + 2|y|/a_n + 1)$$
$$> (a_n/2)(2|y|/a_n)$$
$$= |y|$$
$$\geq y.$$

2. Let

$$f(x) = \begin{cases} \frac{1}{q}, & x = p/q \in \mathbb{Q}, \text{ expressed in lowest form}, \\ 0, & x \in \mathbb{R}\setminus\mathbb{Q} \end{cases}$$

a. Prove that $f$ is discontinuous (not continuous) at every point $a \in \mathbb{Q}$.

**Solution.** Let $a \in \mathbb{Q}$. Write $a = p/q$ in lowest form. Select $\varepsilon = 1/q$. Let $\delta > 0$. By HW2 #7, there is an irrational number $x \in (a, a + \delta)$. For this choice of $x$, we have $|f(a) - f(x)| = 1/q \geq \varepsilon$. We conclude that $f$ is not continuous at $a$.

b. Prove that $f$ is continuous at every point $a \in \mathbb{R}\setminus\mathbb{Q}$.

**Solution.** Let $a \in \mathbb{R}\setminus\mathbb{Q}$. Since $D(f) = \mathbb{R}$, the domain requirement is met. Let $\varepsilon > 0$. Observe that there are at most $1/\varepsilon$ natural numbers $q \in \mathbb{N}$ with $1/q \geq \varepsilon$. For each such number $q$, there are at most $2q \leq 2/\varepsilon$ integers $p$ with $|a - p/q| \leq 1$. For each such pair $(p, q)$, let $r_{p,q} = |a - p/q|$. Note that since $p/q$ is rational and $a$ is irrational, we have that $r_{p,q} > 0$. Let $r$ be the smallest of the numbers $r_{p,q}$; since we are selecting the smallest of at most $2/\varepsilon^2$ positive numbers, we have $r > 0$. Select $\delta = \min(1, r)$.

Let $x \in (a - \delta, a + \delta)$. If $x \in \mathbb{R}\setminus\mathbb{Q}$ then $f(x) = 0$ and thus $|f(a) - f(x)| = |0 - 0| = 0 < \varepsilon$. If $x \in \mathbb{Q}$, then we can write $x = p/q$ in lowest form. We will show that $q > 1/\varepsilon$. Indeed, if $q < 1/\varepsilon$, then the number $r_{p,q}$ was one of the numbers considered above, so we have $r \leq r_{p,q}$, and thus $\delta \leq r_{p,q}$. This means that $|a - x| \geq \delta$, which contradicts the assumption that $x \in (a - \delta, a + \delta)$. We conclude that $q > 1/\varepsilon$. Thus $|f(a) - f(x)| = |0 - 1/q| < \varepsilon$, as desired.

3. Let $f: [a, b] \to \mathbb{R}$ be continuous. Prove that the range of $f$ is a finite length closed interval.

**Solution.** First, by the extreme value theorem there exist numbers $c, d \in [a, b]$ so that

$$f(c) \leq f(x) \leq f(d) \text{ for all } x \in [a, b]. \quad (3)$$

Let $p = f(c)$ and let $q = f(d)$. We will show that $R(f) = [p, q]$. Indeed, (3) implies that $R(f) \subset [p, q]$, and since $f(c) = p$ and $f(d) = q$, we have that $p \in R(f)$ and $q \in R(f)$. It remains to show that $(p, q) \subset R(f)$. Let $y \in (p, q)$. By the intermediate value theorem, there exists a point $x \in [a, b]$ so that $f(x) = y$, and hence $(p, q) \subset R(f)$. 