Math 120 Homework 3

- Due Friday September 27th at 10 am. Homework should be submitted using Canvas.
- **Collaboration Policy**: You are welcome (and encouraged) to work on the homework in groups. However, each student must write up the homework on their own, and must use their own wording (i.e. don't jusy copy the solutions from your friend). If you do collaborate with others, please list the name of your collaborators at the top of the homework.
- You are encouraged (though not required) to type up your solutions. If you choose to do this, I strongly recommend that you use the typesetting software LaTeX. LaTeX is used by the entire mathematics community, and if you intend to go into math, youll need to learn it sooner or later. "The Not So Short Introduction to LaTeX" is a good place to start. This guide can be found at http://tug.ctan.org/info/lshort/english/lshort.pdf . You can also download the .tex source file for this homework and take a look at that.
- Each homework problem should be correct as stated. Occasionally, however, I might screw something up and give you an impossible homework problem. If you believe a problem is incorrect, please email me. If you are right, the first person to point out an error will get +1 on that homework, and I will post an updated version.

Universal Quantifiers

In this section we'll get a bit more experience with the universal quantifiers $\forall$ and $\exists$, and their negations. Here "s.t." means "such that."

1. Write down the negation of each of the following expressions. You do not need to determine whether the expressions are true or false
a) $\forall\ x \in \mathbb{R},\ \exists\ y \in \mathbb{R}$ s.t. $x^2 > y$.
b) $\forall\ x \in \mathbb{R}$ with $x > 0,\ \exists\ y \in \mathbb{R}$ s.t. $x > y^2$.
c) $\forall\ x \in \mathbb{R}$ with $x > 0,\ \exists\ y \in \mathbb{R}$ s.t. $\forall\ z \in \mathbb{R},$ we have $x > y^2 - z$.
d) $\forall\ \epsilon \in \mathbb{R}$ with $\epsilon > 0,\ \exists\ \delta \in \mathbb{R}$ with $\delta > 0$ s.t. $\forall\ x \in \mathbb{R}$ with $|x - 3| < \delta,$ we have $|x^2 - 8| < \epsilon$.

2. For each of the expressions from Question 1, determine whether the expression is true, or its negation is true, and (briefly) explain why.

Limits

3. Using the definition of a limit, prove that the following two statements are true.
a) $\lim_{x\to 2} x^2 + x = 6$.
b) $\lim_{x\to 1} x^3 + x^2 + x = 3$.
Hint: don't forget to verify the domain requirement!

4. Using the definition of a limit, prove that the following two statements are false.
a) $\lim_{x\to 10} 3x = 29$.
b) $\lim_{x\to 1}(x^2/x) = 2$.

5. In lecture, we will discuss the limit rule: If $\lim_{x\to a} f(x) = L$ and $\lim_{x\to a} g(x) = M$ with $M \neq 0$, then

$$\lim_{x\to a}(f/g)(x) = L/M.$$

This question will explore what can happen if $\lim_{x\to a} g(x) = 0$. Note that you do not need to know the above limit rule to do this problem, the limit rule just explains why this problem is relevant.

a) Give an example of functions f, g with $\lim_{x\to a} f(x) = 0$, $\lim_{x\to a} g(x) = 0$, and $\lim_{x\to a}(f/g)(x) = 1$.

b) Give an example of functions f, g with $\lim_{x\to a} f(x) = 0$, $\lim_{x\to a} g(x) = 0$, and $\lim_{x\to a}(f/g)(x) = 0$.

c) Give an example of functions f, g with $\lim_{x\to a} f(x) = 0$, $\lim_{x\to a} g(x) = 0$, and $\lim_{x\to a}(f/g)(x)$ does not exist (as a real number).

In each of the above problems, prove that your answer is correct.

6. Considered the following rather strange function:

$$f(x) = \begin{cases} x, & \text{if } x \in \mathbb{Q}, \\ 0, & \text{if } x \notin \mathbb{Q}. \end{cases}$$

For example, $f(1/2) = 1/2$, while $f(\sqrt{2}) = 0$.

a) Prove that $\lim_{x\to 0} f(x) = 0$.

b) Prove that for every $a \neq 0$, $\lim_{x\to a} f(x)$ does not exist as a real number.