Math 120 Homework 2 Solutions

Upper bounds and least upper bounds

1. Let $S \subset \mathbb{R}$. Prove that if $x \in \mathbb{R}$ is an upper bound for $S$, then $x + 1$ is also an upper bound for $S$.

Solution.
We need to show that for every $y \in S$, we have $y \leq x + 1$. Since $x$ is an upper bound for $S$, we know that for every $y \in S$, we have $y \leq x$. But since $x < x + 1$, we have $y \leq x < x + 1$, and thus $y \leq x + 1$ (indeed, the stronger statement $y < x + 1$ is true, but we don’t need this). Thus $x + 1$ is an upper bound for $S$.

2. Let $S \subset \mathbb{R}$. Suppose that $x \in \mathbb{R}$ and $y \in \mathbb{R}$ are least upper bounds for $S$. Prove that $x = y$.

Solution. Since $x$ is a least upper bound for $S$, and $y$ is an upper bound for $S$, we have $x \leq y$. Similarly, since $y$ is a least upper bound for $S$ and $x$ is an upper bound an upper bound for $S$, we have $y \leq x$. We conclude $x = y$.

3. Let $S \subset \mathbb{R}$. Suppose that $x \in \mathbb{R}$ is an upper bound for $S$. Must it always be true that $x - 1$ is an upper bound for $S$? If so, then prove it. If not, then give an example of a set $S$ and an upper bound $x$ where $x - 1$ is not an upper bound for $S$.

Solution.
No: Let $S = \{1, 2, 3, 4, 5\}$ and let $x = 5$. Then $x$ is an upper bound for $S$, but $x - 1 = 4$ is not an upper bound for $S$ since $5 \in S$ and $5 > (x - 1) = 4$.

The rational numbers do not have the least upper bound property

In lecture, we considered the set

$$S = \{x \in \mathbb{Q} : x > 0, \quad x^2 < 2\}.$$

We proved in lecture that if $z \in \mathbb{Q}$ satisfies $z > 0$ and $z^2 > 2$, then $z$ is an upper bound for $S$. In particular, $S$ is bounded above.

In this problem, we will show that there does not exist a rational number $x \in \mathbb{Q}$ with the property that (A): $x$ is an upper bound for $S$, and (B): if $y \in \mathbb{Q}$ with $y < x$, then $y$ is not an upper bound for $S$. The conclusion is that the set of rational numbers does not have the least upper bound property. This is why we need to work with the real numbers rather than the rational numbers when doing calculus.

4. Let $S$ be defined as above.

a. Prove that if $x$ is a rational number that is greater than zero, then $(2x + 2)/(2 + x)$ is also a rational number.

b. Prove that if $x$ is a rational number that is an upper bound for $S$, then $x > 0$ and $x^2 > 2$. Hint: it might be useful to use the fact that if $a > 0$ and $b > 0$ are real rational numbers with $a > b$, then $a^2 > b^2$ (this is true for real numbers as well, of course).

c. Prove that if $x$ is a rational number that is an upper bound for $S$, then $(2x + 2)/(2 + x)$ is also
an upper bound for $S$.

d. Prove that if $x$ is a rational number that is an upper bound for $S$, then $(2x + 2)/(2 + x) < x$.

**Solution.**
a. If $x \in \mathbb{Q}$ then there exist integers $p$ and $q$ so that $q \neq 0$ and $x = p/q$. Then

$$\frac{2x + 2}{2 + x} = \frac{2(p/q) + 2}{2 + (p/q)} = \frac{(2p + q)/q}{(2q + p)/q} = \frac{2p + q}{p + 2q}.$$  

Note the fact $q \neq 0$ was crucial for this calculation.

b. We will prove this by contradiction. Suppose $x$ is a rational number that is an upper bound for $S$, and either $x \leq 0$ or $x^2 \leq 2$. If $x \leq 0$ then clearly $x$ is not an upper bound for $S$, since $1 \in S$ and $x < 1$. If $x > 0$ and $x^2 \leq 2$, then since there does not exist a rational number $y$ with $y^2 = 2$, we must have $x^2 < 2$. But by the previous problem, $(2x + 2)/(2 + x)$ is also rational; $(2x + 2)/(2 + x) = x + (2 - q^2)/(2 + q) > 0$, and

$$0 < \left(\frac{(2x + 2)}{(2 + x)}\right)^2 = 2 - \frac{2 - x^2}{(2 + x)^2} < 2,$$

so $(2x + 2)/(2 + x) \in S$. Thus $x$ cannot be an upper bound for $S$, since $(2x + 2)/(2 + x) \in S$ and $(2x + 2)/(2 + x) > x$.

c. By part b, if $x$ is an upper bound for $S$ that $x > 0$ and $x^2 > 2$. But this immediately implies that $(2x + 2)/(2 + x) > 0$. Next, we compute

$$\left(\frac{(2x + 2)}{(2 + x)}\right)^2 = 2 + \frac{x^2 - 2}{(2 + x)^2} > 2.$$  

We proved in lecture that if $z > 0$ and $z^2 > 2$, then $z$ is an upper bound for $S$. We conclude that $(2x + 2)/(2 + x)$ is an upper bound for $S$.

d. We have

$$(2x + 2)/(2 + x) = x - \frac{x^2 - 2}{x + 2}.$$  

Since $x^2 > 2$, we conclude that $\frac{x^2 - 2}{x + 2} > 0$ and thus $(2x + 2)/(2 + x) < x$.

**Density**

5. Let $x, y \in \mathbb{R}$ with $y - x > 1$. Prove that there is an integer $z \in \mathbb{Z}$ with $x < z < y$.

**Solution.** For each real number $r \in \mathbb{R}$, define $\lfloor r \rfloor$ to be the largest integer $m \in \mathbb{Z}$ with $m \leq r$. Note that with this definition, $r - 1 < \lfloor r \rfloor \leq r$.

Now, select $z = \lfloor x + 1 \rfloor$. We have that $x = (x + 1) - 1 < z \leq x + 1 < y$, as desired.

6. Let $x, y \in \mathbb{R}$ with $x < y$. Prove that there is a rational number $p/q \in \mathbb{Q}$ with $x < p/q < y$.

**Solution.** Let $t = y - x$. Since $x < y$, we have $t > 0$. Let $q$ be a natural number larger than $1/t$. Then $qy - qx = q(y - x) > (1/t)(y - x) > 1$, so by problem 4, there is an integer $p \in \mathbb{Z}$ with $qx < p < qy$. Dividing by $q$, we conclude $x < p/q < y$, as desired.

7. A number $x \in \mathbb{R}$ is called irrational if it is not rational, i.e. if $x \in \mathbb{R}\setminus\mathbb{Q}$. Using a proof by contradiction, show that if $x$ is irrational, $y$ is rational, and $y \neq 0$, then the product $xy$ is irrational.
Solution. Suppose for contradiction that \( xy \) was rational, i.e. there exists \( p \in \mathbb{Z} \) and \( q \in \mathbb{N} \) with \( xy = p/q \). Since \( y \) is rational, we can write \( y = s/t \) with \( s \in \mathbb{Z} \), \( s \neq 0 \), and \( t \in \mathbb{N} \). Then \( x = p/(qt) = (pt)/(sq) \), which is rational (if \( s < 0 \) then we can write this as \((-pt)/(-sq))\).

8. Let \( x, y \in \mathbb{R} \) with \( x < y \). Prove that there is an irrational number \( z \in \mathbb{R}\setminus\mathbb{Q} \) with \( x < z < y \). Hint: problem 8 from HW 1 might be helpful.

Solution. Since \( \sqrt{2} > 0 \), we have \( x/\sqrt{2} < y/\sqrt{2} \). If \( 0 \in (x/\sqrt{2}, y/\sqrt{2}) \), then by problem 5 there exists a rational number \( a \) with \( 0 < a < y/\sqrt{2} \). Such a number certainly satisfies \( x/\sqrt{2} < a < y/\sqrt{2} \) and \( a \neq 0 \). Alternately, if \( 0 \notin (x/\sqrt{2}, y/\sqrt{2}) \), then by problem 5 there exists a rational number \( a \) with \( x/\sqrt{2} < a < y/\sqrt{2} \). Again, such a number satisfies \( x/\sqrt{2} < a < y/\sqrt{2} \) and \( a \neq 0 \).

Define \( z = a\sqrt{2} \). By problem 6, \( z \) is irrational, and we have \( x < z < y \), as desired.

Injectivity

Definition. We say that a rational number \( m/n \) (with \( m \in \mathbb{Z} \) and \( n \in \mathbb{N} \)) is in lowest form if whenever \( m'/n' \in \mathbb{Z} \) and \( n' \in \mathbb{N} \) with \( m/n = m'/n' \), we have \( n \leq n' \). Equivalently, \( m \) and \( n \) have no common factors.

9. Consider the function \( f: \mathbb{Q} \to \mathbb{R} \) defined as follows: \( f(x) = m/n^2 \), where \( x = m/n \) with \( m/n \) in lowest form. For example, \( f(1/3) = 1/9 \), and \( f(4/5) = 4/25 \). Prove that \( f \) is injective.

Solution. This problem requires some material not covered in lecture. As a result, it has been made a bonus problem. Students who answer it correctly can receive above 100% on the homework.

First, note that if \( x \in \mathbb{Q} \), then there is a unique way to write the number \( x \in \mathbb{Q} \) as a ratio \( m/n \) in lowest form (with \( m \in \mathbb{Z} \) and \( n \in \mathbb{N} \)). Indeed, suppose that \( m/n \) and \( m'/n' \) are two ways of writing \( x \) in lowest form. Then \( n \leq n' \) and \( n' \leq n \), so \( n = n' \). But then \( m = m' \).

Next we will recall a key property of prime numbers: if \( p \) is a prime number, if \( a \) and \( b \) are integers, and if \( p \) divides the product \( ab \), then \( p \) must divide at least one of \( a \) or \( b \). In particular, by setting \( a = b \), if \( p \) divides \( a^2 \) then \( p \) must divide \( a \).

With this fact, we can prove that if \( m/n \) is in lowest form, then \( m/n^2 \) is also in lowest form. To prove this, it suffices to prove the contrapositive: If \( m/n \) is a rational number and if \( m/n^2 \) is not in lowest form, then \( m/n \) is not in lowest form. Indeed, suppose \( m/n^2 \) is not in lowest form. Let \( p \in \mathbb{N} \) be the smallest natural number \( \geq 2 \) that divides both \( m \) and \( n^2 \). \( p \) must be prime, since if any integer \( q \geq 2 \) divides \( p \), then \( q \) would also divide \( m \) and \( n^2 \), which is a contradiction. Since \( p \) is prime and \( p \) divides \( n^2 \), we have that \( p \) divides \( n \), and thus \( p \) divides both \( m \) and \( n \).

We are now ready to prove that \( f \) is injective. Let \( x, x' \in \mathbb{Q} \) with \( f(x) = f(x') \). Write \( x = m/n \) and \( x' = m'/n' \) in lowest form. Then \( m/n^2 = f(x) = f(x') = m'/n'^2 \). Thus \( m/n^2 \) and \( m'/n'^2 \) are two ways of expressing the same rational number in lowest form. But since there is only one way of expressing a rational number in lowest form, we conclude that \( m = m' \) and \( n^2 = (n')^2 \). Since \( n \in \mathbb{N} \) and in particular \( n > 0 \), this implies \( n = n' \), and thus \( x = x' \).