Upper bounds and least upper bounds

1. Let $S \subset \mathbb{R}$. Prove that if $x \in \mathbb{R}$ is an upper bound for $S$, then $x + 1$ is also an upper bound for $S$. (in particular, this shows that if $S$ has an upper bound, then it has infinitely many upper bounds.)

2. Let $S \subset \mathbb{R}$. Suppose that $x \in \mathbb{R}$ and $y \in \mathbb{R}$ are least upper bounds for $S$. Prove that $x = y$. (you’ve just proved that if $S$ has a least upper bound, then this least upper bound is unique.)

3. Let $S \subset \mathbb{R}$. Suppose that $x \in \mathbb{R}$ is an upper bound for $S$. Must it always be true that $x - 1$ is an upper bound for $S$? If so, then prove it. If not, then give an example of a set $S$ and an upper bound $x$ where $x - 1$ is not an upper bound for $S$.

The rational numbers do not have the least upper bound property

In lecture, we considered the set

$$S = \{x \in \mathbb{Q} : x > 0, \ x^2 < 2\}.$$ 

We proved in lecture that if $z \in \mathbb{Q}$ satisfies $z > 0$ and $z^2 > 2$, then $z$ is an upper bound for $S$. In particular, $S$ is bounded above.

In this problem, we will show that there does not exist a rational number $x \in \mathbb{Q}$ with the property that (A): $x$ is an upper bound for $S$, and (B): if $y \in \mathbb{Q}$ with $y < x$, then $y$ is not an upper bound for $S$. The conclusion is that the set of rational numbers does not have the least upper bound property. This is why we need to work with the real numbers rather than the rational numbers when doing calculus.
4. Let $S$ be defined as above.
   a. Prove that if $x$ is a rational number that is greater than zero, then $(2x + 2)/(2 + x)$ is also a rational number.
   b. Prove that if $x$ is a rational number that is an upper bound for $S$, then $x > 0$ and $x^2 > 2$. Hint: it might be useful to use the fact that if $a > 0$ and $b > 0$ are real rational numbers with $a > b$, then $a^2 > b^2$ (this is true for real numbers as well, of course).
   c. Prove that if $x$ is a rational number that is an upper bound for $S$, then $(2x + 2)/(2 + x)$ is also an upper bound for $S$.
   d. Prove that if $x$ is a rational number that is an upper bound for $S$, then $(2x + 2)/(2 + x) < x$.

Density

5. Let $x, y \in \mathbb{R}$ with $y - x > 1$. Prove that there is an integer $z \in \mathbb{Z}$ with $x < z < y$.
6. Let $x, y \in \mathbb{R}$ with $x < y$. Prove that there is a rational number $p/q \in \mathbb{Q}$ with $x < p/q < y$.
7. A number $x \in \mathbb{R}$ is called irrational if it is not rational, i.e. if $x \in \mathbb{R} \setminus \mathbb{Q}$. Using a proof by contradiction, show that if $x$ is irrational, $y$ is rational, and $y \neq 0$, then the product $xy$ is irrational.
8. Let $x, y \in \mathbb{R}$ with $x < y$. Prove that there is an irrational number $z \in \mathbb{R} \setminus \mathbb{Q}$ with $x < z < y$. Hint: problem 8 from HW 1 might be helpful.

Injectivity

Definition. We say that a rational number $p/q$ (with $p \in \mathbb{Z}$ and $q \in \mathbb{N}$) is in lowest form if whenever $p' \in \mathbb{Z}$ and $q' \in \mathbb{N}$ with $p/q = p'/q'$, we have $q \leq q'$. Equivalently, $p$ and $q$ have no common factors.
9. Consider the function $f : \mathbb{Q} \rightarrow \mathbb{R}$ defined as follows: $f(x) = p/q^2$, where $x = p/q$ with $p/q$ in lowest form. For example, $f(1/3) = 1/9$, and $f(4/5) = 4/25$. Prove that $f$ is injective.