Math 120 Homework 2

- Due Friday September 20th at 10 am. Homework should be submitted using Canvas.
- **Collaboration Policy**: You are welcome (and encouraged) to work on the homework in groups. However, each student must write up the homework on their own, and must use their own wording (i.e. don't jusy copy the solutions from your friend). If you do collaborate with others, please list the name of your collaborators at the top of the homework.
- You are encouraged (though not required) to type up your solutions. If you choose to do this, I strongly recommend that you use the typesetting software LaTeX. LaTeX is used by the entire mathematics community, and if you intend to go into math, youll need to learn it sooner or later. "The Not So Short Introduction to LaTeX" is a good place to start. This guide can be found at http://tug.ctan.org/info/lshort/english/lshort.pdf . You can also download the .tex source file for this homework and take a look at that.
- Each homework problem should be correct as stated. Occasionally, however, I might screw something up and give you an impossible homework problem. If you believe a problem is incorrect, please email me. If you are right, the first person to point out an error will get +1 on that homework, and I will post an updated version.

Upper bounds and least upper bounds

1. Let $S \subset \mathbb{R}$. Prove that if $x \in \mathbb{R}$ is an upper bound for S, then $x + 1$ is also an upper bound for S. (in particular, this shows that if S has an upper bound, then it has infinitely many upper bounds.)

2. Let $S \subset \mathbb{R}$. Suppose that $x \in \mathbb{R}$ and $y \in \mathbb{R}$ are least upper bounds for S. Prove that $x = y$. (you've just proved that if S has a least upper bound, then this least upper bound is unique.)

3. Let $S \subset \mathbb{R}$. Suppose that $x \in \mathbb{R}$ is an upper bound for S. Must it always be true that $x - 1$ is an upper bound for S? If so, then prove it. If not, then give an example of a set S and an upper bound x where $x - 1$ is not an upper bound for S.

The rational numbers do not have the least upper bound property

In lecture, we considered the set

$$S = \{x \in \mathbb{Q} \colon x > 0,\ x^2 < 2\}.$$

We proved in lecture that if $z \in \mathbb{Q}$ satisfies $z > 0$ and $z^2 > 2$, then z is an upper bound for S. In particular, S is bounded above.

In this problem, we will show that there does not exist a *rational* number $x \in \mathbb{Q}$ with the property that (A): x is an upper bound for S, and (B): if $y \in \mathbb{Q}$ with $y < x$, then y is not an upper bound for S. The conclusion is that the set of rational numbers does not have the least upper bound property. This is why we need to work with the real numbers rather than the rational numbers when doing calculus.

4. Let S be defined as above.
a. Prove that if x is a rational number that is greater than zero, then $(2x+2)/(2+x)$ is also a rational number.
b. Prove that if x is a rational number that is an upper bound for S, then $x > 0$ and $x^2 > 2$. Hint: it might be useful to use the fact that if $a > 0$ and $b > 0$ are real rational numbers with $a > b$, then $a^2 > b^2$ (this is true for real numbers as well, of course).
c. Prove that if x is a rational number that is an upper bound for S, then $(2x+2)/(2+x)$ is also an upper bound for S.
d. Prove that if x is a rational number that is an upper bound for S, then $(2x+2)/(2+x) < x$.

Density

5. Let $x, y \in \mathbb{R}$ with $y - x > 1$. Prove that there is an integer $z \in \mathbb{Z}$ with $x < z < y$.

6. Let $x, y \in \mathbb{R}$ with $x < y$. Prove that there is a rational number $p/q \in \mathbb{Q}$ with $x < p/q < y$.

7. A number $x \in \mathbb{R}$ is called irrational if it is not rational, i.e. if $x \in \mathbb{R} \backslash \mathbb{Q}$. Using a proof by contradiction, show that if x is irrational, y is rational, and $y \neq 0$, then the product xy is irrational.

8. Let $x, y \in \mathbb{R}$ with $x < y$. Prove that there is an irrational number $z \in \mathbb{R} \backslash \mathbb{Q}$ with $x < z < y$. Hint: problem 8 from HW 1 might be helpful.

Injectivity

Definition. We say that a rational number p/q (with $p \in \mathbb{Z}$ and $q \in \mathbb{N}$) is in *lowest form* if whenever $p' \in \mathbb{Z}$ and $q' \in \mathbb{N}$ with $p/q = p'/q'$, we have $q \leq q'$. Equivalently, p and q have no common factors.
9. Consider the function $f \colon \mathbb{Q} \to \mathbb{R}$ defined as follows: $f(x) = p/q^2$, where $x = p/q$ with p/q in lowest form. For example, $f(1/3) = 1/9$, and $f(4/5) = 4/25$. Prove that f is injective.