Math 120 Homework 10 Solutions

1. Graph the function \( f(x) = \arctan(\tan(x)) \) for all \( x \) in \( D(f) \cap [-4\pi, 4\pi] \). You do not need to prove that your graph is correct, but draw it carefully—mark your \( x \) and \( y \) axes, scale things properly, and be sure to get the domain correct.

\[ \text{Solution.} \]

2. Prove that

\( \sinh(x + y) = \sinh(x) \cosh(y) + \cosh(x) \sinh(y) \),

and

\( \cosh(x + y) = \cosh(x) \cosh(y) + \sinh(x) \sinh(y) \).

\[ \text{Solution. We have} \]

\[ \sinh(x + y) = \frac{1}{2} (e^{x+y} - e^{-(x+y)}) = \frac{1}{2} (e^x e^y - e^{-x} e^{-y}) = \frac{1}{4} ((e^x - e^{-x})(e^y + e^{-y}) + (e^x + e^{-x})(e^y - e^{-y})) = \sinh(x) \cosh(y) + \cosh(x) \sinh(y). \]

A similar computation shows that \( \cosh(x + y) = \cosh(x) \cosh(y) + \sinh(x) \sinh(y) \). (you should do this computation)

3. At which points on the ellipse \( x^2 + 3y^2 = 1 \) is the tangent line parallel to the line \( y = x \)? Prove that your answer is correct.

\[ \text{Solution. The line } y = x \text{ has slope 1. We need to find all point } (x, y) \text{ on the curve } x^2 + 3y^2 = 1 \text{ where the curve has slope 1. If we regard } y = y(x) \text{ as a function of } x, \text{ then we have} \]

\[ 0 = \frac{d}{dx}(x^2 + 3y^2) = 2x + 6y \frac{dy}{dx} \]
re-arranging, we obtain
\[ \frac{dy}{dx} = \frac{-2x}{6y} = -\frac{x}{3y}, \]
whenever \( y \neq 0 \). Solving \(-x/3y = 1, \ x^2 + 3y^2 = 1 \) we obtain the single equation \( 12y^2 = 1 \), or \( y = \pm 1/\sqrt{12} \). The corresponding points are \( (\sqrt{3}/2, -1/\sqrt{12}) \) and \( (-\sqrt{3}/2, 1/\sqrt{12}) \). However, this reasoning is only valid if \( y \neq 0 \). There are two points on the curve \( x^2 + 3y^2 = 1 \) where \( y = 0 \); the points are \( (1, 0) \) and \( (-1, 0) \). At both these points the tangent line to the curve \( x^2 + 3y^2 = 1 \) is vertical, so it is not parallel to the line \( y = x \).

Thus the two points where the ellipse \( x^2 + 3y^2 = 1 \) is the tangent line parallel to the line \( y = x \) are \( (\sqrt{3}/2, -1/\sqrt{12}) \) and \( (-\sqrt{3}/2, 1/\sqrt{12}) \).

4. Compute the slope of the tangent line of the curve \( y^5 + 2xy^3 + 3x^2y + 10x = 16 \) at the point \( (1, 1) \).

Solution. If we regard \( y \) as a function of \( x \) and use implicit differentiation, then
\[
0 = \frac{d}{dx}(y^5 + 2xy^3 + 3x^2y + 10x) = 5y^4 \frac{dy}{dx} + 2y^3 + 6xy^2 \frac{dy}{dx} + 6xy + 3x^2 \frac{dy}{dx} + 10.
\]
re-arranging, we get
\[ \frac{dy}{dx} = \frac{-2y^3 + 6xy + 10}{5y^4 + 6xy^2 + 3x^2} \]
whenever the denominator is non-zero. Plugging in \((x, y) = (1, 1)\), the denominator is \( 5 + 6 + 3 = 14 \neq 0 \), so we can compute
\[ \frac{dy}{dx} = \frac{-2 + 6 + 10}{5 + 6 + 3} = -\frac{9}{7}. \]

5. Use L’Hôpital’s rule to prove the following

(a) Prove that if \( f \) and \( g \) are polynomials with \( f(x) = a_n x^n + \ldots + a_0 \) and \( g(x) = b_n x^n + \ldots + b_0 \), with \( a_n \neq 0, b_n \neq 0 \), then
\[ \lim_{x \to \infty} \frac{f(x)}{g(x)} = \frac{a_n}{b_n}. \]

(b) Prove that if \( f \) and \( g \) are polynomials with \( \deg(g) > \deg(f) \), then
\[ \lim_{x \to \infty} \frac{f(x)}{g(x)} = 0. \]

Solution

a. We will prove the result by induction on \( n \). The base case \( n = 0 \) is trivial: we have \( f(x) = a_0 \) and \( g(x) = a_0 \), so
\[ \lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{a_0}{b_0} = a_0/b_0. \]

Next, suppose the result has been proved for all pairs of polynomials of degree at most \( n \), and let \( f \) and \( g \) be polynomials of degree \( n + 1 \). Since \( \deg f \geq 1 \), the limit rule for quotients of polynomials (proved in lecture) says that \( \lim_{x \to \infty} f(x) = \infty \) or \( -\infty \). Similarly, \( \lim_{x \to \infty} g(x) = \infty \) or \( -\infty \). We proved in lecture that for any polynomial \( g \), there exists a number \( R > 0 \) so that if \( x > R \) then
\( g(x) \neq 0 \). Thus \( \lim_{x \to \infty} \frac{f(x)}{g(x)} \) is of the indeterminate form \( \pm \infty \), and L’Hopital’s rule can be applied. If \( f(x) = a_{n+1}x^{n+1} + a_nx^n + \ldots + a_0 \), then \( f'(x) = (n+1)a_{n+1}x^n + na_nx^{n-1} + \ldots + a_1 \), and similarly if \( g(x) = b_{n+1}x^{n+1} + b_nx^n + \ldots + b_0 \), then \( g'(x) = (n+1)b_{n+1}x^n + nb_nx^{n-1} + \ldots + b_1 \). These are polynomials of degree \( n \), so by the induction assumption,

\[
\lim_{x \to \infty} \frac{f'(x)}{g'(x)} = \frac{(n+1)a_{n+1}}{(n+1)b_{n+1}} = \frac{a_{n+1}}{b_{n+1}}.
\]

Thus by L’Hopital’s rule,

\[
\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)} = \frac{a_{n+1}}{b_{n+1}},
\]

which completes the induction step and finishes the proof.

b. We will again prove the result by induction on the degree of \( f \). If \( f(x) = a_0 \) has degree 0, then since \( g \) has degree > 0, we have \( \lim_{x \to \infty} |g(x)| = \infty \), so \( \lim_{x \to \infty} f(x)/g(x) = \lim_{x \to \infty} a_0/g(x) = 0 \). Now suppose the result has been proved for all pairs of polynomials where the numerator has degree \( \leq n \), and let \( f, g \) be polynomials where \( \deg f = n + 1 \) and \( \deg(g) > \deg(f) \). The same argument from part a shows that we may apply L’hopital’s rule to conclude that

\[
\lim_{x \to \infty} \frac{f(x)}{g(x)} = \lim_{x \to \infty} \frac{f'(x)}{g'(x)} = 0,
\]

where for the second equality we used the induction assumption, since \( \deg f' = n \) and \( \deg g' > \deg f' \). This completes the induction step and concludes the proof.