Math 120 Homework 4

• Due Friday October 13 at start of class.

• If your homework is longer than one page, staple the pages together, and put your name on each sheet of paper. Homework lacking a staple will lose 2 points.

• Each homework problem should be correct as stated. Occasionally, however, I might screw something up and give you an impossible homework problem. If you believe a problem is incorrect, please email me. If you are right, the first person to point out an error will get +1 on that homework, and I will post an updated version.

Recall that in HW 3, you were asked to give an example of functions \( f \) and \( g \) so that \( \lim_{x \to 1} g(x) = 1 \), \( \lim_{x \to 1} f(x) = 1 \), but \( \lim_{x \to 1} f \circ g(x) \neq 1 \).

1. Let \( f \) and \( g \) be functions whose domain is \( \mathbb{R} \). Suppose that \( f(1) = 1 \); \( f \) is continuous at 1, and \( \lim_{x \to 1} g(x) = 1 \). Prove that
   \[
   \lim_{x \to 1} f \circ g(x) = 1.
   \]

2. This problem shows a relationship between one-sided limits and limits at infinity. Let \( g(x) = 1/x \). Let \( f(x) \) be a function and suppose that \( (0, \infty) \subset D(f) \).
   a. Suppose that \( \lim_{x \to 0^+} f(x) = L \). Prove that \( \lim_{x \to \infty} f \circ g(x) = L \) (don’t forget to establish both requirements for the limit to exist).
   b. Suppose that \( \lim_{x \to \infty} f \circ g(x) = L \). Prove that \( \lim_{x \to 0^+} f(x) = L \).

3. Let \( f \) and \( g \) be functions, and suppose \( f \) and \( g \) are continuous (in particular, this means \( D(f) = \mathbb{R}, D(g) = \mathbb{R} \)). Suppose that for every rational number \( x \in \mathbb{Q} \), we have \( f(x) = g(x) \). Prove that \( f(x) = g(x) \) for every number \( x \in \mathbb{R} \).

4 a. Let \( f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0 \) be a polynomial. Suppose that \( a_n > 0 \) and \( a_0 < 0 \). Prove that there exists a point \( x \in \mathbb{R} \) with \( f(x) = 0 \).

b. Give an example of a non-constant polynomial \( f \) (i.e. the polynomial must have degree at least one) so that \( f(x) \neq 0 \) for all \( x \in \mathbb{R} \). You do not have to prove that your example is correct.