Math 120 Homework 3

- Due Friday September 30 at start of class.
- If your homework is longer than one page, **staple** the pages together, and put your name on each sheet of paper. **Homework that is not stapled will loose 1 point.**
- Each homework problem should be correct as stated. Occasionally, however, I might screw something up and give you an impossible homework problem. If you believe a problem is incorrect, please email me. If you are right, the first person to point out an error will get +1 on that homework, and I will post an updated version.

Limits

1. (3 points) Give an example of a function $f(x)$ for which $\lim_{x\to 0^+} f(x) = 1$ exists, but for which the statement $\lim_{x\to 0^-} f(x) = L$ is false for every real number L. Prove that your answer is correct.

2. This problem shows a relationship between one-sided limits and limits at infinity. Let $g(x) = 1/x$. Let $f(x)$ be a function and suppose that $(0,\infty) \subset D(f)$.

a. (4 points) Suppose that $\lim_{x\to 0^+} f(x) = L$. Prove that $\lim_{x\to\infty} f \circ g(x) = L$ (don't forget to establish *both* requirements for the limit to exist).

b. (4 points) Suppose that $\lim_{x\to\infty} f \circ g(x) = L$. Prove that $\lim_{x\to 0^+} f(x) = L$.

In class, we established limit rules for the sum, difference, product, and quotient of limits. These results also hold for one-sided limits. In the case of products, we have:

Theorem 1 (Product rule for one-sided limits) *Let f and g be functions, $a \in \mathbb{R}$, and suppose $\lim_{x\to a^+} f(x) = L_1$, $\lim_{x\to a^+} g(x) = L_2$. Then $\lim_{x\to a^+}(fg)(x) = L_1 L_2$.*

3. (5 points) Suppose that

$$\lim_{x\to\infty} \frac{\sqrt{x+1}}{\sqrt{2x+1}} = L$$

for some real number L (i.e. you may assume that the limit exists and is a real number). Using Problem 2 and the product rule for one-sided limits, prove that

$$\lim_{x\to\infty} \frac{\sqrt{x+1}}{\sqrt{2x+1}} = \frac{1}{\sqrt{2}}.$$

Continuity

4. (6 points) Let f and g be functions, and suppose f and g are continuous (in particular, this means $D(f) = \mathbb{R}$, $D(g) = \mathbb{R}$). Suppose that for every rational number $x \in \mathbb{Q}$, we have $f(x) = g(x)$. Prove that $f(x) = g(x)$ for *every* number $x \in \mathbb{R}$.

5. Define

$$f(x) = \begin{cases} 1/p^2, & \text{if } x = q/p \text{ in lowest form,} \\ 0, & \text{if } x \in \mathbb{R}\backslash\mathbb{Q}. \end{cases}$$

a. (4 points) Prove that if $a \in \mathbb{Q}$, then f is not continuous at a.

b. (4 points) prove that if $a \in \mathbb{R}\backslash\mathbb{Q}$, then f *is* continuous at a.