Math 120 Homework 1

- Due Friday September 16 at start of class.
- If your homework is longer than one page, **staple** the pages together, and put your name on each sheet of paper.
- **Collaboration Policy**: You are welcome (and encouraged) to work on the homework in groups. However, each student must write up the homework on their own, and must use their own wording (i.e. don't jusy copy the solutions from your friend). If you do collaborate with others, please list the name of your collaborators at the top of the homework.
- You are encouraged (though not required) to type up your solutions. If you choose to do this, I strongly recommend that you use the typesetting software LaTeX. LaTeX is used by the entire mathematics community, and if you intend to go into math, youll need to learn it sooner or later. "The Not So Short Introduction to LaTeX" is a good place to start. This guide can be found at http://tug.ctan.org/info/lshort/english/lshort.pdf . You can also download the .tex source file for this homework and take a look at that.
- Each homework problem should be correct as stated. Occasionally, however, I might screw something up and give you an impossible homework problem. If you believe a problem is incorrect, please email me. If you are right, the first person to point out an error will get +1 on that homework, and I will post an updated version.

Sets

1. (3 points) Suppose that $S \cap T = \emptyset$. Prove that $S \backslash T = S$ and $T \backslash S = T$.

2. (3 points) Suppose that $S \subset T$. Prove that $S \backslash T = \emptyset$. Must it be the case that $T \backslash S = \emptyset$?

Definition: If S is a set, we write $|S|$ (called the "cardinality of S") to denote the number of elements of S. For example, if $S = \{\text{red}, \text{ blue}, \text{ green}\}$ then $|S| = 3$, while if $S = \{2, 4, 6, 8, 10\}$, then $|S| = 5$. Some sets have infinitely many elements. In this case, we say $|S| = \infty$. For example, $|\mathbb{Z}| = \infty$.

3. (5 points) Let S and T be sets, and suppose that both $|S|$ and $|T|$ are finite (i.e. neither one is infinity). Prove that

$$|S \cup T| = |S| + |T| - |S \cap T|.$$

Upper bounds and least upper bounds

4. (3 points) Let $S \subset \mathbb{R}$. Prove that if $x \in \mathbb{R}$ is an upper bound for S, then $x + 1$ is also an upper bound for S.

Remark: You don't have to prove this, but the above statement implies that if S has an upper bound, then it has infinitely many upper bounds.

Definition: Let x be an integer. We say that an integer r *divides* x if $r \neq 0$ and x/r is an integer. For example, 3 divides 6, but 3 does not divide 7. (according to this definition, if $x = 0$ then *every* integer divides x)

Definition: An integer x is called *even* if 2 divides x. Otherwise, it is called *odd.*

Definition: Let p and q be integers, neither of which is 0. We say that p and q have a *common divisor* if there is an integer r which is not 1 or -1 such that r divides p and r divides q. For example, $p = 3$ and $q = 6$ have a common divisor, namely 3. On the other hand, $p = 3$ and $q = 7$ do not have a common divisor.

Definition: If $q \neq 0$, we say that the fraction q/p is in *lowest form* if p and q do not have a common divisor and if $p > 0$. If $q = 0$, we say the fraction $0/p$ is in lowest form if $p = 1$.

Remark: In the following problems, you can use the fact that every rational number can be written in a unique way as q/p, where q/p is in lowest form.

5 In this section, we will use a proof by contradiction to prove that $\sqrt{2}$ is not a rational number.
a. (2 points) Prove that if p and q are both even integers and $p \neq 0$, then q/p is *not* in lowest form.
b. (3 points) Prove that if x is an integer and x^2 is even, then x is even (you may have to recall some properties about divisibility from high school. If you do this, clearly state what properties you are using).
c. (2 points) Suppose that $\sqrt{2}$ is a rational number. Then you can write $\sqrt{2} = q/p$, where q/p is in lowest form. Prove that q^2 is even.
d. (2 points) Prove that p must be even (hint: from parts *b* and *c*, you know that q is even, which means we can write $q = 2r$ for some integer r. Try re-arranging the equation $2 = (2r)^2/p^2$).
e. (2 points) Explain why we have arrived at a contradiction to the assumption that $\sqrt{2}$ is a rational number.

6 Consider the set $S = \{y \ \in \mathbb{Q}: \ y^2 < 2\}$.
a. (1 points) Give an example of an upper bound for S.
b. (4 points) Using the least upper bound property for the reals, we can conclude that when we think of S as a subset of $\mathbb{R}$, then S has a least upper bound. In class, we proved that this least upper bound is unique. Prove that this least upper bound is $\sqrt{2}$.
c. (4 points) Let s and t be real numbers with $s < t$. Prove that there is a rational number q/p with $s < q/p < t$.
d. (2 points) Prove that if q/p is a rational number with $q/p < \sqrt{2}$, then q/p is *not* an upper bound for S (hint: use part c)
e. (2 points) Prove that if q/p is a rational number with $q/p > \sqrt{2}$, then q/p is *not* a least upper bound for S (hint: use part c).

Remark: you just proved that the rational numbers do *not* have the least upper bound property. Indeed, if the rational numbers *did* have the least upper bound property, then there would be some rational number q/p that was a least upper bound for S. But in part *d*, you proved that $p/q \geq \sqrt{2}$ and in part *e* you proved that $q/p \leq \sqrt{2}$. Thus we must have $q/p = \sqrt{2}$, but $\sqrt{2}$ is not a rational number; this is a contradiction.