1. Prove that the number $\sqrt[3]{2}$ is irrational.

**Proof:** Assume for a contradiction that $\sqrt[3]{2}$ is rational. Then we see that $\sqrt[3]{2} = \frac{m}{n}$, where $m \in \mathbb{Z}$ and $n \in \mathbb{N}$. Then, since $n \neq 0$, we see $\gcd(m, n)$ exists and is nonzero. Thus, calling $a = m/\gcd(m, n)$ and $b = n/\gcd(m, n)$, we see $\sqrt[3]{2} = \frac{a}{b}$ and moreover, $\gcd(a, b) = 1$. Then, we get, $2 = a^3/b^3$, and hence, $2b^3 = a^3$. This implies that $a^3$ has to be even. Thus, since product of odd numbers is odd, we see that $a$ cannot be odd. Thus, $a$ is even. This implies that $a = 2k$ for some $k \in \mathbb{Z}$. Thus, plugging this into $2b^3 = a^3$, we get $2b^3 = a^3 = (2k)^3 = 8k^3$. Hence, $b^3 = 4k^3$. This implies that $b^3$ is even, and hence $b$ is even. This is a contradiction since if $a, b$ are both even, then $\gcd(a, b) \geq 2$, but we know that $\gcd(a, b) = 1$.

Therefore $\sqrt[3]{2}$ is irrational.

2. Prove that the number $\log_2(3)$ is irrational.

**Proof:** Assume for a contradiction that $\log_2(3)$ is rational. This means that we can write $\log_2(3) = \frac{m}{n}$, where $\min\mathbb{Z}, n \in \mathbb{N}$. Assume moreover that $\gcd(m, n) = 1$ (otherwise, we can do what we have done in question 1). Then we see $2^{m/n} = 3$. This implies, $2^m = 3^n$. Thus, we see that $3 \nmid 2^m$. Moreover, we know that $m > 0$ since $\log_2(3) > 0$. This gives us the contradiction, since $3$ is prime and $3 \nmid 2$, and hence $3 \nmid 2^m$.

Therefore $\log_2(3)$ is irrational.

3. Let $x \in \mathbb{R}$ satisfy $x^7 + 5x^2 - 3 = 0$. Then prove that $x$ is irrational.

**Proof:** Assume for a contradiction that $x$ is rational. This means that we can write $x = \frac{m}{n}$, where $\min\mathbb{Z}, n \in \mathbb{N}$. Assume moreover that $\gcd(m, n) = 1$. Then, plugging this into the equation, we get

$$\frac{m^7}{n^7} + \frac{5m^2}{n^2} - 3 = 0.$$ 

Then multiplying both sides by $n^7$ we get

$$m^7 + 5m^2n^5 - 3n^7 = 0.$$ 

Then, since $\gcd(m, n) = 1$, we know that $m, n$ cannot both be even. Then, we have three cases.

**Case 1:** $m$ is odd, $n$ is even: In this case, we see that $m^7$ is odd, $5m^2n^5$ is even, and $3n^7$ is also even. Then we see that $m^7 + 5m^2n^5 - 3n^7$ is odd, which contradicts with the fact that $m^7 + 5m^2n^5 - 3n^7 = 0$ and $0$ is even.

**Case 2:** $m$ is even, $n$ is odd: In this case, we see that $m^7$ is even, $5m^2n^5$ is even, and $3n^7$ is odd. Then we see that $m^7 + 5m^2n^5 - 3n^7$ is odd, which contradicts with the fact that $m^7 + 5m^2n^5 - 3n^7 = 0$ and $0$ is even.

**Case 1:** $m$ is odd, $n$ is odd: In this case, we see that $m^7$ is odd, $5m^2n^5$ is odd, and $3n^7$ is also odd. Then we see that $m^7 + 5m^2n^5 - 3n^7$ is odd, which contradicts with the fact that $m^7 + 5m^2n^5 - 3n^7 = 0$ and $0$ is even.

Therefore, any real solution of the equation $x^7 + 5x^2 - 3 = 0$ is irrational.
4. Let $a, b, c \in \mathbb{Z}$. If $a^2 + b^2 = c^2$, then $a$ or $b$ is even.

**Proof:** Let $a, b, c \in \mathbb{Z}$ and assume for a contradiction that $a^2 + b^2 = c^2$ and $a$ and $b$ are both odd. Then, we see that $a = 2k + 1$ for some $k \in \mathbb{Z}$ and $b = 2m + 1$ for some $m \in \mathbb{Z}$. Then we see that $a^2 + b^2 = (2k + 1)^2 + (2m + 1)^2 = 4k^2 + 4k + 4m^2 + 4m + 2 = 2(2k^2 + 2m^2 + 2k + 2m + 1) = c^2$. Since, $(2k^2 + 2m^2 + 2k + 2m + 1) \in \mathbb{Z}$, we see that $c^2$ is even, which implies $c$ is even. Thus, $c = 2n$ for some $n \in \mathbb{Z}$. Hence, we get

$$a^2 + b^2 = (2k + 1)^2 + (2m + 1)^2 = 4k^2 + 4k + 4m^2 + 4m + 2 = 4n^2 = c^2,$$

which implies that

$$2 = 4n^2 - (4k^2 + 4k + 4m^2 + 4m) = 4(n^2 - k^2 - m^2 - k - m).$$

Since $(n^2 - k^2 - m^2 - k - m) \in \mathbb{Z}$, this implies $4 \mid 2$, which is a contradiction. Therefore $a$ or $b$ has to be even.

5. Prove that if $k$ is a positive integer and $\sqrt{k}$ is not an integer, then $\sqrt{k}$ is irrational.

**Proof:** Assume for a contradiction that $k$ is a positive integer, $\sqrt{k}$ is not an integer and $\sqrt{k}$ is rational. Then we see that $\sqrt{k} = \frac{m}{n}$, where $\gcd(m, n) = 1$, that is, the ratio $\frac{m}{n}$ is simplified.

Then, by Bézout’s identity we know that $\exists x, y \in \mathbb{Z}$ such that $mx + ny = 1$. Then, multiplying both sides of this equation by $m$, we get $m^2x + nm^2 = m$. Moreover, since $\sqrt{k} = \frac{m}{n}$, we see that $k = \frac{m^2}{n^2}$ which implies $m^2 = kn^2$. Then, using this equality, we get

$$m^2x + nm^2 = n^2kx + nm^2 = n(\frac{m^2}{n^2}x + my) = m.$$

Thus, we see that $n \mid m$. Since we know that $\gcd(m, n) = 1$. This implies that $n = 1$, which means that $\sqrt{k} = m \in \mathbb{Z}$, which contradicts with $\sqrt{k}$ not being an integer.

Therefore the statement is true.

6. Let $n \in \mathbb{N}$, $n \geq 2$, and $a, b, c \in \mathbb{Z}$. Prove that if $ab \equiv 1 \pmod{n}$, then $\forall c \neq 0 \pmod{n}$ we have $ac \not\equiv 0 \pmod{n}$.

**Proof:** Let $n \in \mathbb{N}$, $n \geq 2$, and $a, b, c \in \mathbb{Z}$. Now, assume for a contradiction that $ab \equiv 1 \pmod{n}$ and $\exists c \neq 0 \pmod{n}$ such that $ac \equiv 0 \pmod{n}$. Since $ac \equiv 0 \pmod{n}$, we see $bac \equiv b0 \equiv 0 \pmod{n}$. Moreover, since $ab \equiv 1 \pmod{n}$, we see $abc \equiv 1c \equiv c \pmod{n}$. Therefore combining these two equivalences, we get $c \equiv abc \equiv 0 \pmod{n}$, which is a contradiction, since $c \neq 0 \pmod{n}$.

Therefore if $ab \equiv 1 \pmod{n}$, then $\forall c \neq 0 \pmod{n}$ we have $ac \not\equiv 0 \pmod{n}$.

7. Prove that there do not exist $a, n \in \mathbb{N}$ such that $a^2 + 35 = 7^n$.

**Proof:** Assume for a contradiction that there exist $a, n \in \mathbb{N}$ such that $a^2 + 35 = 7^n$. Then we see that $a^2 = 7(7^{n-1} - 5)$. Then we see $7 \mid a^2$, and since 7 is prime, we see that $7 \mid a$. Thus, $a = 7m$ for some $m \in \mathbb{Z}$. Then plugging this into the original equation, we get $49m^2 + 35 = 7^n$. Then, dividing both sides by 7, we get $7m^2 + 5 = 7^{n-1}$. Now, we have two cases for $n$. If $n = 1$, then we see that the equation becomes $7m^2 + 5 = 1$, which is a contradiction since the left hand side is greater than 5. If $n$ is greater than 1, then $n - 1 > 0$ and hence $7 \mid 7^{n-1}$. Since $7m^2 + 5 = 7^{n-1}$, we see that $7^{n-1} - 7m^2 = 5$. This is also a contradiction since left hand side is divisible by 7 whereas the right hand side is not.

Therefore we see that there do not exist $a, n \in \mathbb{N}$ such that $a^2 + 35 = 7^n$. 
8. Let $A, B$ be nonempty finite sets and assume that there is a bijection, $f$, from $A$ to $B$. Then prove that if $g : A \to B$ is an injective function, then it is surjective. Would this statement be still true if the sets were not finite?

**Proof:** Let $A, B$ be nonempty finite sets and assume that there is a bijection, $f$, from $A$ to $B$. Assume for a contradiction that $g : A \to B$ is an injective function, but it is not surjective. Then, we know that $g(A) \subset B$ and $B - g(A) \neq \emptyset$.

Then we can define the function $\hat{g} : A \to g(A)$ as $\hat{g}(x) = g(x)$ for all $x \in A$. Then, we see that $\hat{g}$ is injective, since $g$ is injective and it is surjective since $\hat{g}(A) = g(A)$. Hence, $\hat{g}$ is a bijective function. Thus, we know that $g^{-1}$ exists and is a bijective function too. Then, we see that $f \circ g^{-1} : g(A) \to B$ is a bijection. This gives us a contradiction since by Pigeonhole Principle, we know that since $|g(A)| < |B|$, there can be no surjection from $g(A)$ to $B$. Hence, we see that $g$ has to be a surjection.

However, this does not work for infinite sets. For a counterexample, we can take $\mathbb{Z}$ and the set of even numbers, call $E$. Then we see that there is a bijection $f : E \to \mathbb{Z}$, defined as $f(x) = \frac{x}{2}$.

9. Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function with $f''(x) > 0$. Prove that $f$ cannot have 3 zeroes.

**Proof:** Let $f : \mathbb{R} \to \mathbb{R}$ be a twice differentiable function with $f''(x) > 0$, and assume for a contradiction that $f$ has 3 distinct zeroes, call them $x_1, x_2, x_3$. WLOG, assume that $x_1 < x_2 < x_3$. Then, we see that $f(x_1) = f(x_2) = 0$, by Rolle’s theorem, $\exists c \in (x_1, x_2)$ such that $f'(c) = 0$. Similarly, since $f(x_2) = f(x_3) = 0$, $\exists d \in (x_2, x_3)$ such that $f'(d) = 0$. We also see that $c \neq d$ since $x_1 < x_2 < x_3$.

Now, since $f'$ is also differentiable and $f'(c) = f'(d)$, again, by Rolle’s theorem, we see that $\exists z \in (c, d)$ such that $f''(z) = 0$. This is a contradiction with the fact that $f''(x) > 0$ for all $x$.

Therefore $f$ cannot have 3 zeroes.

10. Let $(x_n)_{n \in \mathbb{N}}$ be a real sequence. Then, recall that we say $(x_n)$ converges to $L$ if

$$\forall \epsilon > 0, \exists N \in \mathbb{N}, \forall n \geq N, |x_n - L| < \epsilon.$$ 

Prove that if a sequence $(y_n)$ converges, then the limit is unique.

**Proof:** Assume for a contradiction that the sequence $y_n$ converges, but the limit is not unique. This means $\exists L_1, L_2$, $L_1 \neq L_2$ such that $y_n$ converges to $L_1$ and $y_n$ converges to $L_2$. Assume WLOG that $L_1 < L_2$.

Then, by definition

$$\forall \epsilon_1 > 0, \exists N_{\epsilon_1} \in \mathbb{N}, \forall n \geq N_{\epsilon_1}, |y_n - L_1| < \epsilon_1,$$

and

$$\forall \epsilon_2 > 0, \exists N_{\epsilon_2} \in \mathbb{N}, \forall n \geq N_{\epsilon_2}, |y_n - L_2| < \epsilon_2.$$ 

Now, let $\epsilon_1 = \epsilon_2 = \frac{L_2 - L_1}{3}$. Then, since $y_n$ converges to $L_1$, we see that $\exists N_{\epsilon_1} \in \mathbb{N}$ such that $\forall n \geq N_{\epsilon_1}, |y_n - L_1| < \frac{L_2 - L_1}{3}$.

Moreover, since $y_n$ converges to $L_2$, we see $\exists N_{\epsilon_2} \in \mathbb{N}$ such that $\forall n \geq N_{\epsilon_2}, |y_n - L_2| < \frac{L_2 - L_1}{3}$.

Now letting $N = \max(N_{\epsilon_1}, N_{\epsilon_2})$, we see $\forall n \geq N, |y_n - L_1| < \frac{L_2 - L_1}{3}$, and $|y_n - L_2| < \frac{L_2 - L_1}{3}$.

Thus, choosing $m = N + 1$, we get $|y_m - L_1| < \frac{L_2 - L_1}{3}$, and $|y_m - L_2| < \frac{L_2 - L_1}{3}$. 

Then, by triangle and reverse triangle inequalities, we see that $|y_m - L_1| < \frac{L_2 - L_1}{3}$ implies $y_m < \frac{L_2 + 2L_1}{3}$ and $|y_m - L_2| < \frac{L_2 - L_1}{3}$ implies $y_m > \frac{2L_2 + L_1}{3}$. This is a contradiction, since $\frac{2L_2 + L_1}{3} > \frac{L_2 + 2L_1}{3}$.

Therefore if a sequence converges, then the limit is unique.