1. Determine whether the following statements are logically equivalent, using truth tables.

(a) \((\neg P) \lor Q\) and \(P \Rightarrow Q\).
(b) \(P \leftrightarrow Q\) and \((\neg P) \leftrightarrow (\neg Q)\).
(c) \(P \Rightarrow (Q \lor R)\) and \(P \Rightarrow ((\neg Q) \Rightarrow R)\).
(d) \((P \lor Q) \Rightarrow R\) and \((P \Rightarrow R) \land (Q \Rightarrow R)\).
(e) \(P \Rightarrow (Q \lor R)\) and \((Q \land R) \Rightarrow P\).

2. Prove that if \(n \in \mathbb{Z}\) and \(n^2 + 4n + 5\) is odd, then \(n\) is even.

3. Prove that if \(n, m \in \mathbb{Z}\) and \(n^2 + m^2\) is even, then \(n, m\) have the same parity.

4. Prove that if \(n\) is an even integer than \(n = 4k\) or \(n = 4k + 2\) for some integer \(k\).

5. Let \(a \in \mathbb{Z}\). Prove that \(3 \mid 5a\) if and only if \(3 \mid a\).

6. Use the logical equivalences given in class to negate the following sentences

(a) 8 is even and 5 is prime.
(b) If \(n\) is a multiple of 4 and 5, then it is a multiple of 10.
(c) \(3 \leq x \leq 6\).
(d) A real number \(x\) is less than \(-2\) or greater than \(2\) if its square is greater than 4.
(e) If a function \(f\) is differentiable everywhere then whenever \(x \in \mathbb{R}\) is a local maximum of \(f\) we have \(f'(x) = 0\).

7. Assume \(a, b \in \mathbb{Z}\). Prove that if \(ax + by = 1\) for some \(x, y \in \mathbb{Z}\), then \(\gcd(a, b) = 1\).

8. Without using the triangle inequality, prove that if \(x \in \mathbb{R}\), then \(|x + 4| + |x - 3| \geq 7\).

9. Let \(m \in \mathbb{Z}\). Prove that if \(5 \nmid m\), then \(m^2 \equiv 1 \pmod{5}\) or \(m^2 \equiv -1 \pmod{5}\).

10. **Bézout’s identity**: Let \(a, b \in \mathbb{Z}\) such that \(\gcd(a, b) = 1\). Then there exists \(x, y \in \mathbb{Z}\) such that \(ax + by = 1\).

    For example, for \(a = 5\) and \(b = 7\) we can take \(x = 10\) and \(b = -7\).

    Using Bézout’s identity, show that for \(a \in \mathbb{Z}\) and \(p\) prime, if \(a \not\equiv 0 \pmod{p}\) then \(ak \equiv 1 \pmod{p}\) for some \(k \in \mathbb{Z}\).