*This final exam has **8 questions** on **12 pages**, for a total of 100 marks.*

Duration: 2 hours 30 minutes

Full Name (Last, First, All middle names): _____

Student-No: _____ Course Section: _____

Signature: _____

Student Conduct during Examinations

• Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.

• Candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.

• No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no candidate shall be permitted to enter the examination room once the examination has begun.

• Candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.

• Candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
(a) speaking or communicating with other candidates, unless otherwise authorized;
(b) purposely exposing written papers to the view of other candidates or imaging devices;
(c) purposely viewing the written papers of other candidates;
(d) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
(e) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)–(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).

• Candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.

• Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.

• Candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

Question:	1	2	3	4	5	6	7	8	Total
Points:	32	10	10	10	8	10	10	10	100
Score:									

Please read the following points carefully before starting to write.

- Read all the questions carefully before starting to work.
- You should give complete arguments and explanations for all your answers and calculations; answers without justifications will not be marked.
- Continue on the back of the previous page if you run out of space.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)
- You may not leave during the first 30 minutes or final 15 minutes of the exam.

32 marks

1. For Part (a) to Part (g), determine whether the statements are true or false — Put True or False in the boxes. Part (h) is not a True/False type question. Justify your answers.

(a) (4 marks) Let A, B be sets. Then $\mathcal{P}(A \cup B) \subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$.

Solution: False. Take $A = \{1\}, B = \{2\}$. Then $\mathcal{P}(A \cup B) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$, $\mathcal{P}(A) = \{\emptyset, \{1\}\}, \mathcal{P}(B) = \{\emptyset, \{2\}\}$. Now, $\{1, 2\} \notin \mathcal{P}(A), \notin \mathcal{P}(B)$, so $\{1, 2\} \notin \mathcal{P}(A) \cup \mathcal{P}(B)$.

(b) (4 marks) Let R be a relation on the set $A = \{1, 2, 3\}$ defined below

$$R = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 1), (1, 3), (3, 1)\}.$$

Then R is an equivalence relation.

Solution: False. Because transitivity fails: $2R1$ and $1R3$, but 2 does not relate to 3.

(c) (4 marks) There is no smallest positive rational number.

Solution: True. Suppose by way of contradiction that q is the smallest positive rational number. Then $\frac{q}{2}$ is a smaller positive rational number, contradiction.

(d) (4 marks) Let $A_n = (0, \frac{1}{n})$ and $B_n = (-\frac{1}{n}, 0)$ be open intervals for each $n \in \mathbb{N}$. Then

$$\bigcap_{n=1}^{\infty} A_n = \bigcap_{n=1}^{\infty} B_n.$$

Solution: True. Both sides equal the empty set.

(continued on next page)

(e) (4 marks) Let f, g, h be three functions from $\mathbb{R}$ to $\mathbb{R}$. Then $f \circ (g+h) = f \circ g + f \circ h$.

Solution: False. Let $f(x) = x^2, g(x) = h(x) = x$. Then $f \circ (g+h)(x) = f(2x) = 4x^2$ but $f \circ g(x) + f \circ h(x) = x^2 + x^2 = 2x^2$.

(f) (4 marks) Suppose $n \in \mathbb{N}$, $n \geq 2$, and $[a], [b], [c] \in \mathbb{Z}_n$ such that $[c] \neq [0]$. If $[a] \cdot [c] = [b] \cdot [c]$, then $[a] = [b]$.

Solution: False. Suppose $n = 6$, $c = 3$, $a = 2$, and $b = 4$. Then $[c] \neq [0]$ and $[a][c] = [6] = 0 = [12] = [b][c]$, but $[a] \neq [b]$.

(g) (4 marks) Let $a, b, c \in \mathbb{R}$ and $c \geq 0$. If $ab < c$, then $a < \sqrt{c}$ or $b < \sqrt{c}$.

Solution: True; proof by contraposition. Suppose $a \geq \sqrt{c}$ and $b \geq \sqrt{c}$. Then $ab \geq \sqrt{c}^2 = c$.

(h) (4 marks) Let P, Q, R and S be statements. Suppose that P is false and $(R \Rightarrow S) \Leftrightarrow (P \wedge Q)$ is true. Find the truth values of R and S.

Solution: We know if P is false then $P \wedge Q$ is false. Since $(R \Rightarrow S) \Leftrightarrow (P \wedge Q)$ is true, $R \Rightarrow S$ must be false. This means R is true and S is false.

10 marks 2. Show for all integers $n \geq 0$ that $\displaystyle\sum_{k=0}^{n} k3^k = \frac{3^{n+1}(2n-1)+3}{4}$.

Solution: Base Case: $n = 0$. The left hand side is equal to $\sum_{k=0}^{0} k3^k = 0$. The right hand side equals $\frac{3^{0+1}(2(0)-1)+3}{4} = \frac{3(-1)+3}{4} = 0$. Hence the equation is true when $n = 0$.

Inductive Step: $n \geq 0$. Assume that $\sum_{k=0}^{n} k3^k = \frac{3^{n+1}(2n-1)+3}{4}$ for some $n \geq 0$. We will show that $\sum_{k=0}^{n+1} k3^k = \frac{3^{(n+1)+1}(2(n+1)-1)+3}{4} = \frac{3^{n+2}(2n+1)+3}{4}$.

$$\begin{aligned}
\sum_{k=0}^{n+1} k3^k &= \left(\sum_{k=0}^{n} k3^k\right) + (n+1)3^{n+1} \\
&= \frac{3^{n+1}(2n-1)+3}{4} + (n+1)3^{n+1} \\
&= \frac{3^{n+1}(2n-1)+3}{4} + \frac{4(n+1)3^{n+1}}{4} \\
&= \frac{3^{n+1}((2n-1)+4(n+1))+3}{4} \\
&= \frac{3^{n+1}(6n+3)+3}{4} \\
&= \frac{3^{n+2}(2n+1)+3}{4}.
\end{aligned}$$

Hence by induction $\sum_{k=0}^{n} k3^k = \frac{3^{n+1}(2n-1)+3}{4}$ for all $n \geq 0$.

10 marks 3. Prove: For all integers n we have $5 \nmid n^2 - 2$.

Solution: Assume to the contrary that there exists an integer n where $5|(n^2-2)$ so $n^2 - 2 = 5k$ for some $k \in \mathbb{Z}$. By the division algorithm $n = 5q + r$ for some integers q and $0 \leq r < 5$.

$$n^2 - 2 = (5q + r)^2 - 2 = 25q^2 + 10qr + r^2 - 2$$

so

$$\begin{aligned} 25q^2 + 10qr + r^2 - 2 &= 5k \\ r^2 - 2 &= 5k - 25q^2 - 10qr \\ r^2 - 2 &= 5(k - 5q^2 - 2qr). \end{aligned}$$

Thus we must have $5|r^2 - 2$, but we only have a few cases for r.

Case 1: $r = 0$ then $r^2 - 2 = -2$ and $5 \nmid -2$. This case is not possible.

Case 2: $r = 1$ then $r^2 - 2 = -1$ and $5 \nmid -1$. This case is not possible.

Case 3: $r = 2$ then $r^2 - 2 = 2$ and $5 \nmid 2$. This case is not possible.

Case 4: $r = 3$ then $r^2 - 2 = 7$ and $5 \nmid 7$. This case is not possible.

Case 5: $r = 4$ then $r^2 - 2 = 14$ and $5 \nmid 14$. This case is not possible.

Since all cases for r are not possible we have a contradiction. Hence for all integers n we have $5 \nmid n^2 - 2$.

10 marks

4. Prove or disprove: for every $a, n \in \mathbb{N}$ with $n \geq 2$, there exist distinct $k, \ell \in \mathbb{N}$ such that n divides $a^k - a^\ell$.

Solution: If $a = 1$, then we can take $k = 1, l = 2$, and clearly $0 | a^k - a^l$ in this case. Assume now $a \neq 1$. Let $A = \{a^1, a^2, ..., a^{n+1}\}$. Note $|A| = n + 1$ because $a^k \neq a^l$ for $k \neq l$ when $a \neq 1, a \in \mathbb{N} - \{1\}$. Define

$$f : A \to \{0, 1, 2, ..., n - 1\}$$

by sending $x \in A$ to the remainder of x when divided by $n \geq 2$. As the codomain of f has n elements, the Pigeonhole principle implies that f cannot be injective. So there are two different elements in A, say a^k and a^l (so $k \neq l$), s.t.

$$f(a^k) = f(a^l).$$

This means a^k and a^l have the same remainder when divided by n. We can write

$$\begin{aligned} a^k &= nk + r \\ a^l &= nl + r. \end{aligned}$$

It follows that $n | (a^k - a^l)$.

8 marks 5. Let A, B, C be sets. Prove: $A \times C \subseteq B \times C$ if and only if $A \subseteq B$ or $C = \emptyset$.

Solution: Assume $A \times C \subseteq B \times C$. If $C = \emptyset$. Done. If $C \neq \emptyset$, let c be an element in C. For any $a \in A$, we have $(a, c) \in A \times C \subseteq B \times C$. So $(a, c) \in B \times C$, this means $a \in B$. So $A \subseteq B$.

Assume $A \subseteq B$ or $C = \emptyset$. If $C = \emptyset$, then $A \times C = \emptyset = B \times C$. Since $\emptyset$ is a subset of any set, we have $A \times C \subseteq B \times C$. If $C \neq \emptyset$, for any $(a, c) \in A \times C$, $a \in A \subseteq B$, so $a \in B$, so $(a, c) \in B \times C$.

10 marks 6. Suppose that $S \subseteq \mathbb{R}$ is a set defined by

$$S = \{x \in \mathbb{R} : x = m\sqrt{\pi} + n\sqrt{2} \text{ for some } m, n \in \mathbb{Z}\}$$

and S' is a proper subset of S defined by

$$S' = \{x \in S : x = p\sqrt{\pi} + q\sqrt{2} \text{ for some prime numbers } p, q\}.$$

(a) Show that S is countably infinite.

(b) Is there a bijection from S' to S? Justify your answer.

(If needed, you may use the fact that $\pi, \sqrt{2}$ are irrational numbers without proof.)

Solution: (a) Define $f : \mathbb{Z} \times \mathbb{Z} : S$ by

$$f(m, n) = m\sqrt{\pi} + n\sqrt{2}.$$

To see f is injective: Assume $f(m_1, n_1) = f(m_2, n_2)$. Then $(m_1 - m_2)\sqrt{\pi} = (n_2 - n_1)\sqrt{2}$. If $m_1 \neq m_2$ then $\pi = (n_2 - n_1)^2 2/(m_1 - m_2)^2$ would be rational, contradicts π is irrational. So $m_1 = m_2$, hence $n_1 = n_2$, and f is injective. To see f is surjective: for any $x \in S$, by definition of S, there exist $m, n \in \mathbb{Z}$ s.t. $x = m\sqrt{\pi} + n\sqrt{2}$, so $f(m, n) = x$. So f is bijective, and $|S| = |\mathbb{Z} \times \mathbb{Z}| = |\mathbb{N}|$.

(b) Since there are infinitely many primes, the subset $\{x \in S : p\sqrt{\pi} + 3\sqrt{2}, p \text{ is prime}\}$ of S' is infinite, so S' is infinite. Now, an infinite subset of a countably infinite set is countably infinite, so $|S'| = |\mathbb{N}| = |S|$. It follows that there is a bijection from S' to S.

10 marks 7. Let $x \in \mathbb{R}$ satisfy $x^7 + 5x^2 - 3 = 0$. Prove that x is irrational.

Solution: Argue by contradiction. Assume $x = p/q$ where $p, q \in \mathbb{Z}$ and in the reduced form. Then

$$p^7 + 5p^2q^5 - 3q^7 = 0.$$

Since x is in the reduced form, p, q cannot be both even, If both are odd, then each of the 3 terms on LSH is odd, so LHS is odd, cannot be 0. If p even and q odd, then $p^7, 5p^2q^5$ are both even, but $3q^7$ is odd, so LHS is odd, cannot be 0. Similarly, if p odd and q even, then LHS is odd which cannot be 0.

10 marks 8. Let $f : A \to B$ be a function. Prove:

(a) (4 marks) If there is a function $g : B \to A$ such that $g \circ f(x) = x$, for all $x \in A$, then f is injective.

(b) (6 marks) If f is injective, then there is a function $g : B \to A$ such that $g \circ f(x) = x$, for all $x \in A$.

Solution: Assume such g exists. If $f(x) = f(y)$, then $g(f(x)) = g(f(y))$. Thus $x = g \circ f(x) = g \circ f(y) = y$. So f is injective.

The other direction: assume f is injective. For any $y \in B$, either $y \in f(A)$ or $y \notin f(A)$. If $y \in f(A)$, then $y = f(a)$ for some $a \in A$. Since f is injective, a is the only element in A which goes to y under f, in other words, $f^{-1}(\{y\})$ has only 1 element. Define $g : B \to A$ by $g(y) = f^{-1}(\{y\})$ when $y \in f(A)$ and $g(y) = a_0$ for some $a_0 \in A$ when $y \notin f(A)$. Now, because $f(x) \in f(A)$, we have $g \circ f(x) = f^{-1}(\{f(x)\}) = x$.

This page has been left blank for your workings and solutions.