*This midterm has **6 questions** on **7 pages** Duration: 50 minutes*

- Read all the questions carefully before starting to work.
- Give complete arguments and explanations for all your calculations; answers without justifications will not be marked.
- Continue on the back of the previous page if you run out of space.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

Full Name (Last, First): _____

Student Number: _____

Signature: _____

Question:	1	2	3	4	5	6	Total
Points:	12	8	6	8	8	8	50
Score:							

12 marks

1. (a) For each statement (i), (ii), and (iii): decide whether it is true or false (write True or False in the box), and then write the negation of the statement (your answer cannot contain the $\sim$ symbol).

(i) $\forall n \in \mathbb{Z},\ \exists y \in \mathbb{R} - \{0\}$ such that $y^n \leq y$. Answer True

Negation: $\exists n \in \mathbb{Z}$, such that $\forall y \in \mathbb{R} - \{0\}$, $y^n > y$.

(ii) $\exists y \in \mathbb{R} - \{0\}$ such that $\forall n \in \mathbb{Z}$, $y^n \leq y$. Answer True

Negation: $\forall y \in \mathbb{R} - \{0\}$, $\exists n \in \mathbb{Z}$ such that $y^n > y$.

(iii) $\forall x \in \mathbb{R} - \{0\}$, $x \leq \frac{1}{3}$ or $\frac{1}{x} \leq 3$. Answer True

Negation: $\exists x \in \mathbb{R} - \{0\}$, such that $x > \frac{1}{3}$ and $\frac{1}{x} > 3$.

(b) Determine, without proof, whether the following statements are true or false. Write True or False in the boxes.

(i) $(\mathbb{R} \times \mathbb{Z}) \cap (\mathbb{Z} \times \mathbb{R}) = \mathbb{Z} \times \mathbb{Z}$ Answer True

(ii) $(\mathbb{R} \times \mathbb{Z}) \cup (\mathbb{Z} \times \mathbb{R}) = \mathbb{R} \times \mathbb{R}$ Answer False

(c) Let $A = \{\emptyset, 1, 2, \{1\}\}$ and $\mathcal{P}(A)$ be the power set of A. Determine whether the following statements are true or false. Write True or False in the boxes.

(i) $1 \in \mathcal{P}(A)$ Answer False

(ii) $\{1\} \in \mathcal{P}(A)$ Answer True

(iii) $\{\{1\}\} \in \mathcal{P}(A)$ Answer True

(iv) $\emptyset \in \mathcal{P}(A)$ Answer True

8 marks 2. (a) (2 marks) Given that the statement S is true, determine whether

$$\Big(P \Rightarrow (Q \vee R)\Big) \Rightarrow \Big(S \vee \big(\sim(Q \wedge R)\big)\Big)$$

is true or false; and explain your answer.

Solution: We see that the statement $\Big(S \vee \big(\sim (Q \wedge R)\big)\Big)$ is true given that S is true. Moreover, if we have a statement of the form $A \Rightarrow B$, and that B is true, we see that $A \Rightarrow B$ is true regardless A is true or false. Thus we see that, since $\Big(S \vee \big(\sim (Q \wedge R)\big)\Big)$ is true, $\Big(P \Rightarrow (Q \vee R)\Big) \Rightarrow \Big(S \vee \big(\sim (Q \wedge R)\big)\Big)$ is true.

(b) (6 marks) Given that the statement S is true, and that the statement

$$\Big(S \vee \big(\sim(Q \wedge R)\big)\Big) \Leftrightarrow \Big(P \Rightarrow (Q \vee R)\Big) \qquad (**)$$

is false, determine whether the statements P, Q and R are true or false; explain your answer.

Solution: Since S is true, any statement of the form $S \vee T$ is true. Therefore the left-hand side of the given biconditional $(**)$ is true. But we know that the biconditional $(**)$ itself is false; this means that its two substatements have opposite truth values. Since the left-hand side is true, the right-hand side $P \Rightarrow (Q \vee R)$ must be false. The only way an implication can be false is if its hypothesis is true but its conclusion is false. Therefore the hypothesis $\boxed{P \text{ must be True}}$, and the conclusion $Q \vee R$ must be false. Finally, an or-statement is false only if both its substatements are false; therefore $\boxed{Q \text{ and } R \text{ are both False}}$.

6 marks

3. Prove that $\sqrt[5]{2}$ is irrational. You may use (without needing to prove it) the fact that n^5 is even if and only if n is even.

Solution: Proof by contradiction. Suppose that $\sqrt[5]{2}$ is rational. By definition, we can then write $\sqrt[5]{2} = \frac{a}{b}$ for integers a, b with $b \neq 0$; moreover, we may assume $\gcd(a, b) = 1$ (so $\frac{a}{b}$ is fully reduced). Then $2 = \frac{a^5}{b^5}$, or $a^5 = 2b^5$. So a^5 is even, therefore a is even. Now we can write $a = 2k$ for some $k \in \mathbb{Z}$. This leads to $(2k)^5 = 2b^5$, which gives $b^5 = 2(2^3k^5)$. In particular, b^5 is even, hence b is even. But now both a and b are even, so $\gcd(a, b) \geq 2$, contradicting the assumption that $\gcd(a, b) = 1$. This contradiction shows that $\sqrt[5]{2}$ is irrational. q.e.d.

8 marks 4. Let $a, b \in \mathbb{Z}$. Prove that $(a-3)b^2$ is odd if and only if a is even and b is odd.

Solution: First we prove: if a is even and b is odd, then $(a-3)b^2$ is odd. Write $a = 2m$ and $b = 2n+1$ for some integers m and n. Then $(a-3)b^2 = (2m-3)(2n+1)^2 = 2(4mn^2 + 4mn + m - 6n^2 - 6n - 2) + 1$ is odd (since $4mn^2 + 4mn + m - 6n^2 - 6n - 2$ is also an integer).

Next we prove: if $(a-3)b^2$ is odd, then a is even and b is odd. We do this by proving the contrapositive: if a is odd or b is even, then $(a-3)b^2$ is even.

Case 1: a is odd. Write $a = 2k+1$ for some integer k; then $(a-3)b^2 = (2k+1-3)b^2 = 2((k-1)b^2)$ is even (since $(k-1)b^2$ is also an integer).

Case 2: b is even. Write $b = 2k$ for some integer k; then $(a-3)b^2 = (a-3)(2k)^2 = 2((a-3)2k^2)$ is even (since $(a-3)2k^2$ is also an integer).

In either case, $(a-3)b^2$ is even. q.e.d.

8 marks 5. Prove: If $n \equiv 1 \pmod 2$ and $m \equiv 3 \pmod 4$, then $n^2 + m \equiv 0 \pmod 4$.

Solution: Since $n \equiv 1 \pmod 2$, $2|(n-1)$. There is $k \in \mathbb{Z}$ such that $n - 1 = 2k$. So $n = 2k + 1$. Since $m \equiv 3 \pmod 4$, $4|(m-3)$. There is $l \in \mathbb{Z}$ such that $m - 3 = 4l$, so $m = 4l + 3$. Now, $n^2 + m = (2k+1)^2 + (4l+3) = 4(k^2 + k + l + 1)$. Therefore, $4|(n^2 + m)$ as $k^2 + k + l + 1 \in \mathbb{Z}$. So $n^2 + m \equiv 0 \pmod 4$. q.e.d.

8 marks

6. Let $n \in \mathbb{Z}$. Show that $2 \mid (n^4 - 5)$ if and only if $4 \mid (n^2 + 7)$. (You can use without proof the fact that ab is odd if and only if a and b are both odd.)

Solution: Since $2|(n^4-5)$, $n^4-5 = 2k$ for some $k \in \mathbb{Z}$. So $n^4 = 2k+5 = 2(k+2)+1$ is odd. Since $n^4 = nn^3$, so n is odd by the Fact, and then $n = 2l+1$ for some $l \in \mathbb{Z}$. Now $n^2 + 7 = (2l+1)^2 + 7 = 4(l^2 + l + 2)$, so $4|(n^2+7)$ as $l^2 + l + 2 \in \mathbb{Z}$.

Now, assume $4|(n^2+7)$. Then $n^2+7 = 4p$ for some $p \in \mathbb{Z}$. So $n^2 = 4p-7 = 2(2p-4)+1$ is odd as $2p - 4 \in \mathbb{Z}$. So $n^4 = n^2n^2$ is odd, and $n^4 - 5$ is even (odd - odd is even). We have $2|(n^4 - 5)$. q.e.d.