Homework 8

1. (Chapter 10: Question 8) If $n \in \mathbb{N}$, then

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}.$$

Proof. We will prove this by using induction on n.

Base step: When $n = 1$ the left hand side is $\frac{1}{2!} = \frac{1}{2}$. When $n = 1$ the right hand side is $1 - \frac{1}{(1+1)!} = \frac{1}{2}$ which proves the equality is true when $n = 1$.

Inductive step: Let $n \geq 1$. We will assume that $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$. Our goal is to show that $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n+1}{((n+1)+1)!} = 1 - \frac{1}{((n+1)+1)!}$. Using our inductive assumption we have

$$\begin{aligned}
\left(\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!}\right) + \frac{n+1}{((n+1)+1)!} &= \left(1 - \frac{1}{(n+1)!}\right) + \frac{n+1}{(n+2)!} \\
&= 1 - \left(\frac{1}{(n+1)!} - \frac{n+1}{(n+2)!}\right) \\
&= 1 - \frac{1}{(n+1)!}\left(1 - \frac{n+1}{n+2}\right) \\
&= 1 - \frac{1}{(n+1)!}\left(\frac{1}{n+2}\right) \\
&= 1 - \frac{1}{((n+1)+1)!}.
\end{aligned}$$

By induction we have thus shown that $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$ for all natural numbers n. $\square$

2. (Chapter 10: Question 12) For any integer $n \geq 0$, it follows that $9|(4^{3n} + 8)$.

Proof. We will do induction on n.

Base step: For $n = 0$, $4^{3\cdot 0} + 8 = 4^0 + 8 = 9$ is divisible by 9.

Inductive step: Assume $9|(4^{3n} + 8)$ for $n \geq 0$. This means $4^{3n} + 8 = 9p$ for some $p \in \mathbb{Z}$. Then

$$4^{3(n+1)} + 8 = 4^3 \cdot 4^{3n} + 8 = 64 \cdot 4^{3n} + 8 = 63 \cdot 4^{3n} + (4^{3n} + 8) = 9(7 \cdot 4^{3n} + p).$$

Therefore, $9|(4^{3(n+1)} + 8)$. By Mathematical Induction $9|(4^{3n} + 8)$ holds for any integer $n \geq 0$. $\square$

3. (Chapter 10: Question 18) Suppose $A_1, A_2, \ldots, A_n$ are sets in some universal set U, and $n \geq 2$. Prove that

$$\overline{A_1 \cup A_2 \cup \cdots \cup A_n} = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n}.$$

Proof. We will prove this by using induction on n.

Base step: $n = 2$. We know whenever we have two sets A_1 and A_2 in the universal set U by DeMorgan's law $\overline{A_1 \cup A_2} = \overline{A_1} \cap \overline{A_2}$.

Inductive step: Let $n \geq 2$. We will assume that give any n subsets $B_1, B_2, \ldots, B_n$ of U that $\overline{B_1 \cup B_2 \cup \cdots \cup B_n} = \overline{B_1} \cap \overline{B_2} \cap \cdots \cap \overline{B_n}$. Our goal is to show that given any $n+1$ subsets $A_1, A_2, \ldots, A_{n+1}$ of U that $\overline{A_1 \cup A_2 \cup \cdots \cup A_{n+1}} = \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_{n+1}}$. So let $A_1, A_2, \ldots, A_{n+1}$ be any $n+1$ sets which are subsets of the universal set U. Note that then $A_1, A_2, \ldots, A_n$ is a collection of n subsets of U so

we can apply our inductive assumption to this collection of sets. Using DeMorgan's law (page 145 in textbook, Exercise 11)and our inductive assumption we have that

$$\begin{aligned}\overline{A_1 \cup A_2 \cup \cdots \cup A_n \cup A_{n+1}} &= \overline{(A_1 \cup A_2 \cup \cdots \cup A_n) \cup A_{n+1}} \\ &= \overline{(A_1 \cup A_2 \cup \cdots \cup A_n)} \cap \overline{A_{n+1}} \\ &= \overline{A_1} \cap \overline{A_2} \cap \cdots \cap \overline{A_n} \cap \overline{A_{n+1}}\end{aligned}$$

which proves our statement by induction. □

4. (Chapter 10: Question 22) If $n \in \mathbb{N}$, then

$$\left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{8}\right)\left(1-\frac{1}{16}\right)\cdots\left(1-\frac{1}{2^n}\right) \geq \frac{1}{4}+\frac{1}{2^{n+1}}.$$

Proof. We will prove this by using induction on n.

Base step: When $n=1$ the left hand side is $(1-\frac{1}{2})=\frac{1}{2}$. When $n=1$ the right hand side is $\frac{1}{4}+\frac{1}{2^{1+1}}=\frac{1}{2}$ and since $\frac{1}{2}\geq\frac{1}{2}$ the inequality is true for $n=1$.

Inductive step: Let $n \geq 1$. We will assume that $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{8}\right)\left(1-\frac{1}{16}\right)\cdots\left(1-\frac{1}{2^n}\right) \geq \frac{1}{4}+\frac{1}{2^{n+1}}$. Our goal is to show that $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{8}\right)\left(1-\frac{1}{16}\right)\cdots\left(1-\frac{1}{2^{n+1}}\right) \geq \frac{1}{4}+\frac{1}{2^{(n+1)+1}}$. Using our inductive assumption we have

$$\begin{aligned}\left[\left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{8}\right)\left(1-\frac{1}{16}\right)\cdots\left(1-\frac{1}{2^n}\right)\right]\left(1-\frac{1}{2^{n+1}}\right) &\geq \left[\frac{1}{4}+\frac{1}{2^{n+1}}\right]\left(1-\frac{1}{2^{n+1}}\right) \\ &= \frac{1}{4}+\frac{1}{2^{n+1}}-\frac{1}{4}\frac{1}{2^{n+1}}-\frac{1}{2^{n+1}}\frac{1}{2^{n+1}} \\ &= \frac{1}{4}+\frac{1}{2^{n+1}}\left(1-\frac{1}{4}-\frac{1}{2^{n+1}}\right) \\ &= \frac{1}{4}+\frac{1}{2^{n+1}}\left(\frac{3}{4}-\frac{1}{2^{n+1}}\right).\end{aligned}$$

Because $n \geq 1$ and $2^{1+1} \leq 2^{n+1}$ we know $\frac{1}{2^{n+1}} \leq \frac{1}{2^{1+1}} = \frac{1}{4}$ so

$$\begin{aligned}\frac{1}{4}+\frac{1}{2^{n+1}}\left(\frac{3}{4}-\frac{1}{2^{n+1}}\right) &\geq \frac{1}{4}+\frac{1}{2^{n+1}}\left(\frac{3}{4}-\frac{1}{4}\right) \\ &= \frac{1}{4}+\frac{1}{2^{n+1}}\left(\frac{1}{2}\right) \\ &= \frac{1}{4}+\frac{1}{2^{(n+1)+1}}\end{aligned}$$

Thus we have shown $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{8}\right)\left(1-\frac{1}{16}\right)\cdots\left(1-\frac{1}{2^{n+1}}\right) \geq \frac{1}{4}+\frac{1}{2^{(n+1)+1}}$. Hence by induction $\left(1-\frac{1}{2}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{8}\right)\left(1-\frac{1}{16}\right)\cdots\left(1-\frac{1}{2^n}\right) \geq \frac{1}{4}+\frac{1}{2^{n+1}}$ for all $n \in \mathbb{N}$. □

5. The *Fibonacci numbers* are defined to be $F_1 = 1, F_2 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n > 2$. Show that for all $k \in \mathbb{N}$, F_{4k} is a multiple of 3.

Proof. We will prove this by inducting on k.

Base step: When $k=1$ we have $m=4k=4$ and $F_m = F_4 = F_3+F_2 = (F_2+F_1)+F_2 = (1+1)+1 = 3$ which is a multiple of 3.

Inductive step: Let $k \geq 1$ we will assume that F_{4k} is a multiple of 3 and show that $F_{4(k+1)}$ is also a multiple of 3. Since F_{4k} is a multiple of 3 we know $F_{4k} = 3x$ for some integer x. Using the Fibonacci recurrence we have

$$\begin{aligned}
F_{4(k+1)} &= F_{4k+4} \\
&= F_{4k+3} + F_{4k+2} \\
&= (F_{4k+2} + F_{4k+1}) + (F_{4k+1} + F_{4k}) \\
&= ((F_{4k+1} + F_{4k}) + F_{4k+1}) + (F_{4k+1} + F_{4k}) \\
&= 3F_{4k+1} + 2F_{4k} \\
&= 3F_{4k+1} + 2(3x) \\
&= 3(F_{4k+1} + 2x).
\end{aligned}$$

Hence $F_{4(k+1)}$ is also a multiple of 3. Therefore, by induction we can conclude that F_{4k} is a multiple of 3 for all $k \in \mathbb{N}$. $\square$

6. Use induction to prove that if A is a finite set of cardinality $n \geq 0$ then $|\mathcal{P}(A)| = 2^n$.

Proof. We proceed by induction on n.

Base case: If $n = 0$, then $A = \emptyset$. Then $\mathcal{P}(A) = \{\emptyset\}$. Thus $|A| = 0$ and $|\mathcal{P}(A)| = 1 = 2^0$. Hence the statement is true for $n = 0$.

Inductive step: Assume that the statement is true for any set of cardinality k. Let B be a set with $|B| = k + 1$. We write B as $B = \{b_1, ..., b_k, b_{k+1}\}$. For any subset C of B, we consider two cases:

- If $b_{k+1} \notin C$, then $C \subseteq \{b_1, ..., b_k\}$; it follows from the induction hypothesis that there are 2^k such subsets.
- If $b_{k+1} \in C$, then $C = D \cup \{b_{k+1}\}$ for some $D \subseteq \{b_1, ..., b_k\}$; again, by the induction hypothesis there are 2^k such sets D.

Since these two cases give disjoint collections of sets, there must be $2^k + 2^k = 2^{k+1}$ subsets of B. Now by induction the statement is true for all $n \geq 0$. $\square$

7. Let $f(x) = x \ln x$ and $x > 0$. Denote $f^{(n)}(x)$ the nth derivative of $f(x)$ for $n \in \mathbb{N}$. Prove that for all integer $n \geq 3$ it holds

$$f^{(n)}(x) = (-1)^n \frac{(n-2)!}{x^{n-1}}.$$

Proof. We use induction on n.

Base case: $n = 3$. Direct computation shows:

$$\begin{aligned}
f'(x) &= 1 + \ln x \\
f''(x) &= \frac{1}{x} \\
f^{(3)}(x) &= -\frac{1}{x^2} = (-1)^3 \frac{(3-2)!}{x^{3-1}}.
\end{aligned}$$

Thus the statement is true for $n = 3$.

Inductive step: Assume $k \geq 3$ and

$$f^{(k)}(x) = (-1)^k \frac{(k-2)!}{x^{k-1}}.$$

For $k+1$, we differentiate the above equality

$$f^{(k+1)}(x) = (f^{(k)}(x))' = (-1)^k(k-2)!(k-1)(-1)x^{-(k-1)-1)} = (-1)^{k+1}(k-1)!x^{-k} = (-1)^{k+1}\frac{((k+1)-2)!}{x^{(k+1)-1}}.$$

By induction the statement is true for all integer $n \geq 3$. $\square$

8. A sequence $a_1, a_2, ..., a_n, ...$ is defined by

$$a_1 = 1, a_2 = 4, \text{ and } a_n = 2a_{n-1} - a_{n-2} + 2 \text{ for all } n \geq 3.$$

Show: $a_n = n^2$ for all $n \in \mathbb{N}$.

Proof. We proceed by strong induction. Since $a_1 = 1 = 1^2$ and $a_2 = 4 = 2^2$, the formula holds for $n = 1, 2$. Assume $a_k = k^2$ for every integer k with $2 \leq k \leq n$. As $n + 1 \geq 3$, we can use the iteration formula as follows

$$\begin{aligned} a_{n+1} &= 2a_n - a_{n-1} + 2 \\ &= 2n^2 - (n-1)^2 + 2 \\ &= 2n^2 - (n^2 - 2n + 1) + 2 \\ &= n^2 + 2n + 1 \\ &= (n+1)^2. \end{aligned}$$

By strong induction, $a_n = n^2$ for all $n \in \mathbb{N}$. $\square$

9. Suppose you begin with a pile of n stones ($n \geq 2$) and split this pile into n piles of one stone each by successively splitting a pile of stones into two smaller piles. Each time you split a pile you multiply the number of stones in each of the two smaller piles you form, so that if these piles have p and q stones in them, respectively, you compute pq. Show that no matter how you split the piles (eventually into n piles of one stone each), the sum of the products computed at each step equals $n(n-1)/2$. (Hint: use strong induction on n.)

Proof. We proceed by strong induction.

Base case: When $n = 2$, the stones are split into two piles of one stone each. Thus $p = q = 1$ and $pq = 1 = (2)(2-1)/2$. So the statement is true for $n = 2$.

Inductive step: Assume that the statement is true for the game of any l stones with $2 \leq l \leq n$. For $n+1$ stones, suppose that we finish the first round of the splitting and have two piles now: one with k stones and one with $n+1-k$ stones, $k > 0$.

Case 1: If $k = 1$ then the pile with $k = 1$ stone will not be changed in the later part of the game and the pile with $n+1-k = n$ falls in the range of the induction hypothesis, so the sum of the products from each step to the end of the game is

$$1 \cdot n + \frac{n(n-1)}{2} = \frac{(n+1)n}{2}.$$

Case 2: If $k = n$, we have a pile with n stones and a pile with 1 stone, so we have the same formula for the sum of the products as in Case 1.

Case 3: If $2 \leq k < n$, we can apply the induction hypothesis to both piles since $2 \leq k \leq n$ and $2 \leq n+1-k \leq n$ (to see this: as $k < n$, $k+1 \leq n$, so $1 \leq n-k$, then $2 \leq n-k+1$; adding n to both sides of $1-k \leq 0$ yields $n+1-k \leq n$). Now, the two piles together contribute a product $k(n+1-k)$,

and individually the two piles contribute $k(k-1)/2$ and $(n+1-k)(n-k)/2$, respectively, according to the induction hypothesis; therefore the total sum is

$$\begin{aligned}
& \frac{k(k-1)}{2} + \frac{(n+1-k)(n-k)}{2} + k(n+1-k) \\
= \; & \frac{1}{2}\left((k^2-k) + ((n+1)n - (n+1)k - kn + k^2) + 2k(n+1-k)\right) \\
= \; & \frac{(n+1)n}{2}.
\end{aligned}$$

Hence, by strong induction, the statement holds for any integer $n \geq 2$. . $\square$