1. *Below is a "proof" that contains an error. Please identify the error, and prove or disprove the original statement.*

   Statement: Let a, b be integers and $a - b$ odd. Then, a and b are of different parity.

   Proof: We prove by cases:

   - a is odd, and b is even: Then, there exists $m, n \in \mathbb{Z}$ such that $a = 2m + 1$ and $b = 2n$. Thus, $a - b = (2m + 1) - (2n) = 2(m - n) + 1$, which is odd,
   - a is even, and b is odd: Then, there exists $m, n \in \mathbb{Z}$ such that $a = 2m$ and $b = 2n + 1$. So, $a - b = 2m - (2n + 1) = 2(m - n - 1) + 1$, which is odd.

   Since in both cases, $a - b$ is odd, we conclude that if $a - b$ is odd, then a and b are of different parity.

2. Find the power set $\mathcal{P}(\{a, b, \{a\}\})$ by listing all its elements.

3. Assume $A \cup B = \{1, 2, 3, \ldots, 10\}$, $A = \{1, 2, 3, 4, 5\}$ and $|A \cap B| = 2$. Find $|B|$.

4. Express each of the following statements as a conditional statement in "if-then" form. For (a),(b) and (c) also write the negation (without phrases like "it is false that"), converse and contrapositive. Your final answers should use clear English, not logical symbols.

   (a) Every odd number is prime.
   (b) Passing the test requires solving all the problems.
   (c) Being first in line guarantees getting a good seat.
   (d) I get mad whenever you do that.
   (e) I won't say that unless I mean it.

5. Show that $(P \wedge R) \wedge (Q \vee R) \equiv (P \wedge R)$. (We have several ways of doing this including using known equivalences, proving the corresponding biconditional or writing out a truth table.)

6. Given a real number x,

   - let $A(x)$ be the statement "$\frac{1}{2} < x < \frac{5}{2}$",
   - let $B(x)$ be the statement "$x \in \mathbb{Z}$",
   - let $C(x)$ be the statement "$x^2 = 1$", and

   Which statements below are true for all $x \in \mathbb{R}$?

   (a) $A(x) \Rightarrow C(x)$
   (b) $C(x) \Rightarrow B(x)$
   (c) $(A(x) \wedge B(x)) \Rightarrow C(x)$
   (d) $C(x) \Rightarrow (A(x) \wedge B(x))$
   (e) $(A(x) \vee C(x)) \Rightarrow B(x)$

7. Consider the following two statements:

   (a) There exist integers a and b such that both $ab < 0$ and $a + b > 0$.
   (b) For all real numbers x and y, $x \neq y$ implies that $x^2 + y^2 > 0$.

   - Using quantifiers, express in symbols the negations of the statements in both (a) and (b).

- Express in words the negations of the statements in (a) and (b).
- Decide which is true in each case, the statement or its negation.

8. Consider the following two statements:

   (a) For all $w \in \mathbb{R}$, there exists $x \in \mathbb{R}$ such that $w < x$.

   (b) There exists $y \in \mathbb{R}$ such that for all $z \in \mathbb{R}$, $y < z$

   One of the statements is true, and the other one is false. Determine which is which and prove your answers (both of them). *(Final exam 2005)*

9. (a) Prove that $3|2n \Leftrightarrow 3|n$.
   Hint. The contrapositive will really help in one direction.

   (b) Prove that if $2|n$ and $3|n$ then $6|n$. *Hint. Consider n modulo* 6.

   (c) Prove that the product of any three consecutive natural numbers is divisible by 6.

10. Is it true that if a natural number is divisible by 4 and by 6, then it must be divisible by $4 \times 6 = 24$?

11. Prove that for any integer n, n is even if and only if n^3 is even.

12. Prove that a sum of a rational number and an irrational number must be irrational.

13. Prove that $2^{1/3}$ is irrational.

14. Prove that $\log_2 3$ is irrational.