

Solutions 3

1. **Question 2.1.10:** In this question we are going to decide whether or not the expression " $(\mathbb{R} \times \mathbb{N}) \cap (\mathbb{N} \times \mathbb{R}) = \mathbb{N} \times \mathbb{N}$ " is a statement, and in case it is decide whether or not it is true.

To be able to do this question, we need to understand what the expression means. The statement reads as: "The intersection of the Cartesian product of \mathbb{R} and \mathbb{N} with the Cartesian product of \mathbb{N} and \mathbb{R} is the Cartesian product of \mathbb{N} and \mathbb{N} ". Now, we see that this is actually a statement.

Moreover,

$$\begin{aligned} (\mathbb{R} \times \mathbb{N}) \cap (\mathbb{N} \times \mathbb{R}) &= \{(x, y) : (x, y) \in \mathbb{R} \times \mathbb{N} \text{ and } (x, y) \in \mathbb{N} \times \mathbb{R}\} \\ &= \{(x, y) : x \in \mathbb{R} \text{ and } x \in \mathbb{N}; \text{ and } y \in \mathbb{N} \text{ and } x \in \mathbb{R}\} \\ &= \{(x, y) : x \in \mathbb{R} \cap \mathbb{N} \text{ and } y \in \mathbb{R} \cap \mathbb{N}\} \\ &= \{(x, y) : x \in \mathbb{N} \text{ and } y \in \mathbb{N}\} \text{ (since } \mathbb{N} \text{ is a subset of } \mathbb{R}\text{)} \\ &= \mathbb{N} \times \mathbb{N}. \end{aligned}$$

Therefore this statement is true.

2. **Question 2.2.12:** In this question we are going to express the sentence "Happy families are all alike, but each unhappy family is unhappy in its own way", in symbolic logic notation.

An easy answer to this question is " $P \wedge Q$ " for the statements " P : Happy families are all alike", and " Q : Each unhappy family is unhappy in its own way". But since we are learning to express more complicated sentences in terms of symbolic logic, we can rewrite this sentence in a more complex way as follows:

Let the open statements P and Q is defined as: " $P(X, a)$: The family X is happy because of a ", and " $Q(X, a)$: The family X is unhappy because of a " then the statement above becomes,

"For all X, Y , and a , $(P(X, a) \wedge P(Y, a)) \wedge (Q(X, b) \wedge \sim(Q(Y, b)))$ for some b ". (Here we assumed that something that doesn't make you happy doesn't always have to make you unhappy).

3. **Question 2.3.2:** In this question we are going to convert the sentence "For a function to be continuous, it is sufficient that it is differentiable", in the form "if P , then Q ".

Recall that "it is sufficient that it is differentiable" means: "whenever it is differentiable". So, our sentence can be written as "A function is continuous, whenever it is differentiable", or equivalently, "A function is continuous, if it is differentiable".

Therefore if we let P and Q to be the sentences " $P(f)$: The function, f , is differentiable" and " $Q(f)$: The function, f , is continuous", then our sentence becomes "If $P(f)$, then $Q(f)$ (for every f)" or equivalently " $P(f) \Rightarrow Q(f)$ (for every f)" (recall that a conditional statement given in terms of open sentences should be seen as a statement that holds for every variable the open sentences take values for.)

4. **Question 2.3.6:** In this question we are going to convert the sentence "Whenever a surface has only one side, it is non-orientable", in the form "if P , then Q ".

Again, if we let " $P(S)$: The surface S has only one side" and " $Q(S)$: The surface S is non-orientable". Then our sentence becomes:

"If $P(S)$, then $Q(S)$ (for every surface S)", or equivalently, " $P(S) \Rightarrow Q(S)$ (for every surface S)".

5. **Question 2.4.4:** In this question we are going to convert the sentence "If $a \in \mathbb{Q}$ then $5a \in \mathbb{Q}$, and if $5a \in \mathbb{Q}$ then $a \in \mathbb{Q}$ ", into a sentence having the form " P if and only if Q ".

The sentence can be converted as: " $a \in \mathbb{Q}$ if and only if $5a \in \mathbb{Q}$ ".

Also, if we let $P(a)$ be the open sentence " $P(a) : a \in \mathbb{Q}$ ", then the sentence above becomes " $P(a)$ if and only if $P(5a)$ (for every a)", or equivalently " $P(a) \Leftrightarrow P(5a)$ (for every a)".

6. **Question 2.5.10:** We are given in this question that the statement $((P \wedge Q) \vee R) \Rightarrow (R \vee S)$ is false and we need to find the truth values of P , Q , R and S .

We see that this is a conditional statement and we know that conditional statements are false only when the “if” statement is true but the “then” statement is false. So this tells us that $((P \wedge Q) \vee R)$ must be true and $(R \vee S)$ must be false. Now we can pay attention to the false “or” statement. We know that an “or” statement is false only when both of the statements that it contains are false. This tells us that for $(R \vee S)$ to be false, both R and S must be false. Now we have to figure out the truth values for P and Q . We know that R is false and we also know that $((P \wedge Q) \vee R)$ is true. This means that $P \wedge Q$ must be true. But we know that $P \wedge Q$ is true when both of the statements P and Q must be true.

Therefore we found out that P and Q are true and R and S are false.

7. **Question 2.6.10:** In this question we are trying to decide whether or not the statements $(P \Rightarrow Q) \vee R$ and $\sim((P \wedge \sim Q) \wedge \sim R)$, are logically equivalent. For that, all we need to do is to look at their truth tables:

P	Q	R	$\sim Q$	$\sim R$	$P \Rightarrow Q$	$(P \Rightarrow Q) \vee R$	$P \wedge \sim Q$	$(P \wedge \sim Q) \wedge \sim R$	$\sim((P \wedge \sim Q) \wedge \sim R)$
T	T	T	F	F	T	T	F	F	T
T	T	F	F	T	T	T	F	F	T
T	F	T	T	F	F	T	T	F	T
F	T	T	F	F	T	T	F	F	T
T	F	F	T	T	F	F	T	T	F
F	T	F	F	T	T	T	F	F	T
F	F	T	T	F	T	T	F	F	T
F	F	F	T	T	T	T	F	F	T

8. **Question 2.7.4:** In this question we are going to write the following sentence as an English sentence:

“ $\forall X \in \mathcal{P}(\mathbb{N}), X \subseteq \mathbb{R}$ ”.

To do this all we have to do is to remember what the symbols meant. So, if we read this statement, it says:

“For all X in the power set of \mathbb{N} , X is a subset of \mathbb{R} ”, which means:

“For all X which is a subset of natural numbers, X is a subset of real numbers”, or, as:

“Every subset of natural numbers is a subset of real numbers”. This is a true statement.

9. **Question 2.7.6:** We are going to translate the following sentence into English:

“ $\exists n \in \mathbb{N}, \forall X \in \mathcal{P}(\mathbb{N}), |X| < n$ ”.

Again, all we have to do is to remember what the symbols meant and be careful with the order of them. So, our statement reads as:

“There exists an n , an element of natural numbers such that for all X in power set of natural numbers size of X is less than n ”, or equivalently as:

“There exists a natural number n such that for every subset X of natural numbers, the size of X is less than n ”, which means:

“There exists a natural number n such that every subset of natural numbers has size less than n ”. This statement is not a true statement since for any number $n \in \mathbb{N}$ given we can construct the set $X = \{1, 2, 3, \dots, (n + 1)\}$ which has size $n + 1$.

10. **Question 2.9.6:** We are going to translate the sentence “For every positive number ϵ , there is a positive number M for which $|f(x) - b| < \epsilon$ whenever $x > M$ ” into symbolic logic.

One thing we have to pay attention here is that the expression “whenever” actually refers to a conditional statement, it says: “if $x > M$, then $|f(x) - b| < \epsilon$ ”. Now we can write it in symbolic logic notation.

“ $\forall \epsilon > 0, \exists M$ s.t. $x > M \Rightarrow |f(x) - b| < \epsilon$ ”.

11. **Question 2.9.12:** In this question we are going to translate the following sentence into symbolic logic: “You can fool some of the people all of the time, and you can fool all of the people some of the time, but you cannot fool all the people all of the time.”

Let $P(x, y)$ be the open sentence, “ $P(x, y)$: You can fool the person x at time y ”. Then then the sentence becomes:

“ $(\exists x, \forall y, P(x, y)) \wedge (\forall x, \exists y, P(x, y)) \wedge \sim(\forall x, \forall y, P(x, y))$ ”, and if we apply the negation:

“ $(\exists x, \forall y, P(x, y)) \wedge (\forall x, \exists y, P(x, y)) \wedge (\exists x, \exists y, \sim P(x, y))$ ”.

If we interpret the second part of the sentence as: “there are times that you can fool everyone”, it may also lead to:

“ $(\exists x, \forall y, P(x, y)) \wedge (\exists y, \forall x, P(x, y)) \wedge (\exists x, \exists y, \sim P(x, y))$ ”.

12. **Question 2.10.2:** In this question we want to negate the sentence “If x is prime, then \sqrt{x} is not a rational number”. We know that if we have a statement of the form “ $P \Rightarrow Q$ ”, then its negation is “ $P \wedge \sim(Q)$ ”. But we have to be careful. The conditional statement in the question is given in terms of open sentences “ $P(x)$: x is prime” and “ $Q(x)$: \sqrt{x} is rational”, which means that this conditional statement should hold for every prime number x .

Thus, the correct form of this statement in symbolic logic is: “ $\forall x, x \text{ is prime} \Rightarrow \sqrt{x} \text{ is not rational}$ ”, or “ $\forall x, x \text{ is prime} \Rightarrow \sqrt{x} \notin \mathbb{Q}$ ”. Therefore its negation is “ $\exists x$ such that x is prime and \sqrt{x} is rational”, or “ $\exists x$ such that x is prime and $\sqrt{x} \in \mathbb{Q}$ ”

13. **Question 2.10.4:** Here we need to negate the sentence “For every positive ϵ there is a positive number δ such that $|x - a| < \delta$ implies $|f(x) - f(a)| < \epsilon$ ”.

Before we negate this statement let’s write it in symbolic logic form:

“ $\forall \epsilon > 0, \exists \delta > 0$ such that $\forall x$ with $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$ ”. So, to negate this sentence all we have to do is to flip the quantifiers and negate the final implication, i.e.:

“ $\sim(\forall \epsilon > 0, \exists \delta > 0$ such that $\forall x$ with $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \epsilon$)”

\Leftrightarrow

“ $\exists \epsilon > 0, \forall \delta > 0, \exists x, \text{ s.t. } (|x - a| < \delta) \wedge (|f(x) - f(a)| \geq \epsilon)$ ”, which read as: “There exists a positive ϵ such that for all positive δ , there exists an x such that $|x - a| < \delta$ and $|f(x) - f(a)| \geq \epsilon$ ”.

14. **Question 2.10.8:** We want to negate the sentence “if x is a rational number and $x \neq 0$, then $\tan(x)$ is not a rational number.”

Again, as in the question 2.10.2, this statement is stated for every x . This means that if we write it in symbolic logic notation we get:

“ $\forall x, x \in \mathbb{Q} - \{0\} \Rightarrow \tan(x) \notin \mathbb{Q}$ ”. Now the negation becomes easier to see.

“ $\sim(\forall x, x \in \mathbb{Q} - \{0\} \Rightarrow \tan(x) \notin \mathbb{Q})$ ” \Leftrightarrow “ $\exists x, (x \in \mathbb{Q} - \{0\}) \wedge (\tan(x) \in \mathbb{Q})$ ”, which reads as: “There is a nonzero rational number x whose tangent is a rational number”.