**Theorem 2.10.** *Let u be harmonic in Ω and let Ω′ be any compact subset of Ω. Then for any multi-index* $\alpha$ *we have*

$$\sup_{\Omega'} |D^\alpha u| \leqslant \left(\frac{n|\alpha|}{d}\right)^{|\alpha|} \sup_{\Omega} |u| \tag{2.32}$$

*where* $d = \operatorname{dist}(\Omega', \partial\Omega)$.

An immediate consequence of the bound (2.32) is the equicontinuity on compact subdomains of the derivatives of any bounded set of harmonic functions. Consequently by Arzela's theorem, we see that any bounded set of harmonic functions forms a *normal family*; that is, we have:

**Theorem 2.11.** *Any bounded sequence of harmonic functions on a domain Ω contains a subsequence converging uniformly on compact subdomains of Ω to a harmonic function.*

The previous convergence theorem, Theorem 2.8, would also follow immediately from Theorem 2.11.

## 2.8. The Dirichlet Problem; the Method of Subharmonic Functions

We are in a position now to approach the question of existence of solutions of the classical Dirichlet problem in arbitrary bounded domains. The treatment here will be accomplished by *Perron's method of subharmonic functions* [PE] which relies heavily on the maximum principle and the solvability of the Dirichlet problem in balls. The method has a number of attractive features in that it is elementary, it separates the interior existence problem from that of the boundary behaviour of solutions, and it is easily extended to more general classes of second order elliptic equations. There are other well known approaches to existence theorems such as the method of integral equations, treated for example in the books [KE 2] [GU], and the variational or Hilbert space approach which we describe in a more general context in Chapter 8.

The definition of $C^2(\Omega)$ subharmonic and superharmonic function is generalized as follows. A $C^0(\Omega)$ function $u$ will be called *subharmonic* (*superharmonic*) in $\Omega$ if for every ball $B \subset\subset \Omega$ and every function $h$ harmonic in $B$ satisfying $u \leqslant (\geqslant) h$ on $\partial B$, we also have $u \leqslant (\geqslant) h$ in $B$. The following properties of $C^0(\Omega)$ subharmonic functions are readily established:

(i) If $u$ is subharmonic in a domain $\Omega$, it satisfies the strong maximum principle in $\Omega$; and if $v$ is superharmonic in a bounded domain $\Omega$ with $v \geqslant u$ on $\partial\Omega$, then either $v > u$ throughout $\Omega$ or $v \equiv u$. To prove the latter assertion, suppose the contrary. Then at some point $x_0 \in \Omega$ we have

$$(u-v)(x_0) = \sup_{\Omega} (u-v) = M \geqslant 0,$$

and we may assume there is a ball $B=B(x_0)$ such that $u-v\not\equiv M$ on $\partial B$. Letting $\bar{u}$, $\bar{v}$ denote the harmonic functions respectively equal to $u$, $v$ on $\partial B$ (Theorem 2.6), one sees that

$$M \geqslant \sup_{\partial B}(\bar{u}-\bar{v}) \geqslant (\bar{u}-\bar{v})(x_0) \geqslant (u-v)(x_0)=M,$$

and hence the equality holds throughout. By the strong maximum principle for harmonic functions (Theorem 2.2) it follows that $\bar{u}-\bar{v}\equiv M$ in $B$ and hence $u-v\equiv M$ on $\partial B$, which contradicts the choice of $B$.

(ii) Let $u$ be subharmonic in $\Omega$ and $B$ be a ball strictly contained in $\Omega$. Denote by $\bar{u}$ the harmonic function in $B$ (given by the Poisson integral of $u$ on $\partial B$) satisfying $\bar{u}=u$ on $\partial B$. We define in $\Omega$ the *harmonic lifting* of $u$ (in $B$) by

$$U(x)=\begin{cases}\bar{u}(x), & x\in B\\ u(x), & x\in\Omega-B.\end{cases} \tag{2.33}$$

Then the function $U$ is also subharmonic in $\Omega$. For consider an arbitrary ball $B'\subset\subset\Omega$ and let $h$ be a harmonic function in $B'$ satisfying $h\geqslant U$ on $\partial B'$. Since $u\leqslant U$ in $B'$ we have $u\leqslant h$ in $B'$ and hence $U\leqslant h$ in $B'-B$. Also since $U$ is harmonic in $B$, we have by the maximum principle $U\leqslant h$ in $B\cap B'$. Consequently $U\leqslant h$ in $B'$ and $U$ is subharmonic in $\Omega$.

(iii) Let $u_1, u_2, \ldots, u_N$ be subharmonic in $\Omega$. Then the function $u(x)=\max\{u_1(x),\ldots,u_N(x)\}$ is also subharmonic in $\Omega$. This is a trivial consequence of the definition of subharmonicity. Corresponding results for superharmonic functions are obtained by replacing $u$ by $-u$ in properties (i), (ii) and (iii).

Now let $\Omega$ be bounded and $\varphi$ be a bounded function on $\partial\Omega$. A $C^0(\bar{\Omega})$ subharmonic function $u$ is called a *subfunction* relative to $\varphi$ if it satisfies $u\leqslant\varphi$ on $\partial\Omega$. Similarly a $C^0(\bar{\Omega})$ superharmonic function is called a *superfunction* relative to $\varphi$ if it satisfies $u\geqslant\varphi$ on $\partial\Omega$. By the maximum principle every subfunction is less than or equal to every superfunction. In particular, constant functions $\leqslant\inf_{\partial\Omega}\varphi$ ($\geqslant\sup_{\partial\Omega}\varphi$) are subfunctions (superfunctions). Let $S_\varphi$ denote the set of subfunctions relative to $\varphi$. The basic result of the Perron method is contained in the following theorem.

**Theorem 2.12** *The function* $u(x)=\sup_{v\in S_\varphi} v(x)$ *is harmonic in* $\Omega$.

*Proof.* By the maximum principle any function $v\in S_\varphi$ satisfies $v\leqslant\sup\varphi$, so that $u$ is well defined. Let $y$ be an arbitrary fixed point of $\Omega$. By the definition of $u$, there exists a sequence $\{v_n\}\subset S_\varphi$ such that $v_n(y)\to u(y)$. By replacing $v_n$ with $\max(v_n, \inf\varphi)$, we may assume that the sequence $\{v_n\}$ is bounded. Now choose $R$ so that the ball $B=B_R(y)\subset\subset\Omega$ and define $V_n$ to be the harmonic lifting of $v_n$ in $B$ according to (2.33). Then $V_n\in S_\varphi$, $V_n(y)\to u(y)$ and by Theorem 2.11 the sequence $\{V_n\}$ contains a subsequence $\{V_{n_k}\}$ converging uniformly in any ball $B_\rho(y)$ with $\rho<R$ to a function $v$ that is harmonic in $B$. Clearly $v\leqslant u$ in $B$ and $v(y)=u(y)$. We claim now that in fact $v=u$ in $B$. For suppose $v(z)<u(z)$ at some $z\in B$. Then there exists



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a function $\bar{u} \in S_\varphi$ such that $v(z) < \bar{u}(z)$. Defining $w_k = \max(\bar{u}, V_{n_k})$ and also the harmonic liftings $W_k$ as in (2.33), we obtain as before a subsequence of the sequence $\{W_k\}$ converging to a harmonic function $w$ satisfying $v \leqslant w \leqslant u$ in $B$ and $v(y) = w(y) = u(y)$. But then by the maximum principle we must have $v = w$ in $B$. This contradicts the definition of $\bar{u}$ and hence $u$ is harmonic in $\Omega$. □

The preceding result exhibits a harmonic function which is a prospective solution (called the *Perron solution*) of the classical Dirichlet problem: $\Delta u = 0$, $u = \varphi$ on $\partial\Omega$. Indeed, if the Dirichlet problem is solvable, its solution is identical with the Perron solution. For let $w$ be the presumed solution. Then clearly $w \in S_\varphi$ and by the maximum principle $w \geqslant u$ for all $u \in S_\varphi$. We note here also that the proof of Theorem 2.12 could have been based on the Harnack convergence theorem, Theorem 2.9, instead of the compactness theorem, Theorem 2.11; (see Problem 2.10).

In the Perron method the study of boundary behaviour of the solution is essentially separate from the existence problem. The continuous assumption of boundary values is connected to the geometric properties of the boundary through the concept of *barrier* function. Let $\xi$ be a point of $\partial\Omega$. Then a $C^0(\bar{\Omega})$ function $w = w_\xi$ is called a *barrier* at $\xi$ relative to $\Omega$ if:

(i) $w$ is superharmonic in $\Omega$;
(ii) $w > 0$ in $\bar{\Omega} - \xi$; $w(\xi) = 0$.

An important feature of the barrier concept is that it is a local property of the boundary $\partial\Omega$. Namely, let us define $w$ to be a *local barrier* at $\xi \in \partial\Omega$ if there is a neighborhood $N$ of $\xi$ such that $w$ satisfies the above definition in $\Omega \cap N$. Then a barrier at $\xi$ relative to $\Omega$ can be defined as follows. Let $B$ be a ball satisfying $\xi \in B \subset\subset N$ and $m = \inf_{N-B} w > 0$. The function

$$\bar{w}(x) = \begin{cases} \min(m, w(x)), & x \in \bar{\Omega} \cap B \\ m, & x \in \bar{\Omega} - B \end{cases}$$

is then a barrier at $\xi$ relative to $\Omega$, as one sees by confirming properties (i) and (ii). Indeed, $\bar{w}$ is continuous in $\bar{\Omega}$ and is superharmonic in $\Omega$ by property (iii) of subharmonic functions; property (ii) is immediate.

A boundary point will be called *regular* (with respect to the Laplacian) if there exists a barrier at that point.

The connection between the barrier and boundary behavior of solutions is contained in the following.

**Lemma 2.13.** *Let $u$ be the harmonic function defined in $\Omega$ by the Perron method (Theorem 2.12). If $\xi$ is a regular boundary point of $\Omega$ and $\varphi$ is continuous at $\xi$, then $u(x) \to \varphi(\xi)$ as $x \to \xi$.*

*Proof.* Choose $\varepsilon > 0$, and let $M = \sup|\varphi|$. Since $\xi$ is a regular boundary point, there is a barrier $w$ at $\xi$ and, by virtue of the continuity of $\varphi$, there are constants $\delta$ and $k$ such that $|\varphi(x) - \varphi(\xi)| < \varepsilon$ if $|x - \xi| < \delta$, and $kw(x) \geqslant 2M$ if $|x - \xi| \geqslant \delta$. The functions

$\varphi(\xi)+\varepsilon+kw$, $\varphi(\xi)-\varepsilon-kw$ are respectively superfunction and subfunction relative to $\varphi$. Hence from the definition of $u$ and the fact that every superfunction dominates every subfunction, we have in $\Omega$.

$$\varphi(\xi)-\varepsilon-kw(x)\leqslant u(x)\leqslant \varphi(\xi)+\varepsilon+kw(x)$$

or

$$|u(x)-\varphi(\xi)|\leqslant \varepsilon+kw(x).$$

Since $w(x)\to 0$ as $x\to\xi$, we obtain $u(x)\to\varphi(\xi)$ as $x\to\xi$. □

This leads immediately to

**Theorem 2.14.** *The classical Dirichlet problem in a bounded domain is solvable for arbitrary continuous boundary values if and only if the boundary points are all regular.*

*Proof.* If the boundary values $\varphi$ are continuous and the boundary $\partial\Omega$ consists of regular points, the preceding lemma states that the harmonic function provided by the Perron method solves the Dirichlet problem. Conversely, suppose that the Dirichlet problem is solvable for all continuous boundary values. Let $\xi\in\partial\Omega$. Then the function $\varphi(x)=|x-\xi|$ is continuous on $\partial\Omega$ and the harmonic function solving the Dirichlet problem in $\Omega$ with boundary values $\varphi$ is obviously a barrier at $\xi$. Hence $\xi$ is regular, as are all points of $\partial\Omega$. □

The important question remains: For what domains are the boundary points regular? It turns out that general sufficient conditions can be stated in terms of local geometric properties of the boundary. We mention some of these conditions below.

If $n=2$, consider a boundary point $z_0$ of a bounded domain $\Omega$ and take the origin at $z_0$ with polar coordinates $r, \theta$. Suppose there is a neighborhood $\mathcal{N}$ of $z_0$ such that a single valued branch of $\theta$ is defined in $\Omega\cap N$, or in a component of $\Omega\cap N$ having $z_0$ on its boundary. One sees that

$$w=-\mathrm{Re}\,\frac{1}{\log z}=-\frac{\log r}{\log^2 r+\theta^2}$$

local barrier at $z_0$ and hence $z_0$ is a regular point. In particular, $z_0$ is a regular ndary point if it is the endpoint of a simple arc lying in the exterior of $\Omega$. Thus Dirichlet problem in the plane is always solvable for continuous boundary in a (bounded) domain whose boundary points are each accessible from the by a simple arc. More generally, the same barrier shows that the boundary roblem is solvable if every component of the complement of the domain of more than a single point. Examples of such domains are domains by a finite number of simple closed curves. Another is the unit disc slit arc; in this case the boundary values can be assigned on opposite sides of

For higher dimensions the situation is substantially different and the Dirichlet problem cannot be solved in corresponding generality. Thus, an example due to Lebesgue shows that a closed surface in three dimensions with a sufficiently sharp inward directed cusp has a non-regular point at the tip of the cusp; (see for example [CH]).

A simple sufficient condition for solvability in a bounded domain $\Omega \subset \mathbb{R}^n$ is that $\Omega$ satisfy the *exterior sphere condition*; that is, for every point $\xi \in \partial\Omega$, there exists a ball $B = B_R(y)$ satisfying $\bar{B} \cap \bar{\Omega} = \xi$. If such a condition is fulfilled, then the function $w$ given by

$$(2.34) \qquad w(x) = \begin{cases} R^{2-n} - |x-y|^{2-n} & \text{for } n \geq 3 \\ \log \dfrac{|x-y|}{R} & \text{for } n = 2 \end{cases}$$

will be a barrier at $\xi$. Consequently the boundary points of a domain with $C^2$ boundary are all regular points; (see Problem 2.11).

## 2.9. Capacity

The physical concept of *capacity* provides another means of characterizing regular and exceptional boundary points. Let $\Omega$ be a bounded domain in $\mathbb{R}^n (n \geq 3)$ with smooth boundary $\partial\Omega$, and let $u$ be the harmonic function (often called the *conductor potential*) defined in the complement of $\bar{\Omega}$ and satisfying the boundary conditions $u = 1$ on $\partial\Omega$ and $u = 0$ at infinity. The existence of $u$ is easily established as the (unique) limit of harmonic functions $u'$ in an expanding sequence of bounded domains having $\partial\Omega$ as an inner boundary (on which $u' = 1$) and with outer boundaries (on which $u' = 0$) tending to infinity. If $\Sigma$ denotes $\partial\Omega$ or any smooth closed surface enclosing $\Omega$, then the quantity

$$(2.35) \qquad \operatorname{cap} \Omega = -\int_{\Sigma} \frac{\partial u}{\partial \nu} \, ds = \int_{\mathbb{R}^n - \Omega} |Du|^2 \, dx \qquad \nu = \text{outer normal}$$

is defined to be the capacity of $\Omega$. In electrostatics, cap $\Omega$ is within a constant factor the total electric charge on the conductor $\partial\Omega$ held at unit potential (relative to infinity).

Capacity can also be defined for domains with nonsmooth boundaries and for any compact set as the (unique) limit of the capacities of a nested sequence of approximating smoothly bounded domains. Equivalent definitions of capacity can be given directly without use of approximating domains (e.g., see [LK]). In particular, we have the variational characterization

$$(2.36) \qquad \operatorname{cap} \Omega = \inf_{v \in K} \int |Dv|^2,$$