Example 1: Find the points on the surface 
\[ xy - z^2 + 1 = 0 \]
that are closest to the origin.

Solution: \[ \min \ w = x^2 + y^2 + z^2 \]
subject to \[ xy - z^2 + 1 = 0 \]

Lagrange multiplier:
\[ \begin{align*}
xy - z^2 + 1 &= 0 \\
2x &= \lambda y \\
2y &= \lambda x \\
2z &= -\lambda z
\end{align*} \]

From (2) & (3) \[ x^2 = y^2 \Rightarrow \text{either } x = y \text{ or } x = -y \]

Case 1: \[ x = y \]
From (4) \[ \text{either } z = 0 \text{ or } \lambda = 1 \]
If \( z = 0 \), then \[ xy - z^2 + 1 = 0 \Rightarrow x^2 + 1 = 0 \], not possible

If \( \lambda = 1 \), then go back to (2) \[ 2x = x \Rightarrow x = 0 \]. So \( y = x = 0 \)
go back to (1) \[ z = \pm 1 \]
We obtain two critical points \((0, 0, \pm 1)\)
Case 2. \( x = -y \)

From 4 \( \Rightarrow z = 0 \) or \( \lambda = 1 \)

If \( z = 0 \), then \( xy - z^2 + 1 = 0 \) \( \Rightarrow -x^2 + 1 = 0 \) \( \Rightarrow x = \pm 1 \)

two critical points \((\pm 1, \mp 1, 0)\)

If \( \lambda = 1 \), then go back to 4 \( \Rightarrow 2x = -x \Rightarrow x = 0 \).
So \( y = 0 \)

go back to 4 \( \Rightarrow z = \pm 1 \)

two critical points \((0, 0, \pm 1)\)

All together we get 4 critical points

\((0, 0, \pm 1)\), \((\pm 1, \mp 1, 0)\)

So

\[
\min w = \min \left( w(0, 0, \pm 1), w(\pm 1, \mp 1, 0) \right)
\]

\[
= \min (1, 2) = 2
\]

\[
\min \text{ distance is } d = \sqrt{\min w} = \sqrt{2}.
\]