MATH 516-101, 2018-2019 Homework Five
Due Date: November 30, 2018

1. Let $u \in H^1(R^n)$ have compact support and be a weak solution of the semilinear PDE

$$-\Delta u + u^3 = f \text{ in } R^n$$

where $f \in L^2$. Prove that $u \in H^2(R^n)$.

Hint: mimic the proof of H^2-estimates but without the cut-off function.

2. Let u be a weak sub-solution of

$$-\sum_{i,j} \partial_{x_j}(a^{ij}\partial_{x_i}u) + c(x)u = f$$

where $\theta \leq (a^{ij}) \leq C_2 < +\infty$. Suppose that $c(x) \in L^{\frac{n}{2}}(B_1), f \in L^q(B_1)$ where $q > \frac{n}{2}$. Show that there exists a generic constant $\epsilon_0 > 0$ such that if $\int_{B_1} |c|^{\frac{n}{2}} dx \leq \epsilon_0$, then

$$\sup_{B_{1/2}} u^+ \leq C(\|u^+\|_{L^2(B_1)} + \|f\|_{L^q(B_1)})$$

Hint: following the Moser's iteration procedure.

3. Show that $u = \log|x|$ is in $H^1(B_1)$, where $B_1 = B_1(0) \subset R^3$ and that it is a weak solution of

$$-\Delta u + c(x)u = 0$$

for some $c(x) \in L^{\frac{3}{2}}(B_1)$ but u is not bounded.

4. Let $u \in H_0^1(\Omega)$ be a weak solution of

$$-\Delta u = |u|^{q-1}u \ \text{ in } \Omega; u = 0 \text{ on } \partial\Omega$$

where $q < \frac{n+2}{n-2}$. Without using Moser's iteration Lemma, use the $W^{2,p}-$ theory only to show that $u \in L^\infty$.

5. Let u be a smooth solution of $Lu = -\sum_{i,j} a^{ij}u_{x_i x_j} = 0$ in U and a^{ij} are C^1 and uniformly elliptic. Set $v := |Du|^2 + \lambda u^2$. Show that

$$Lv \leq 0 \ \text{ in } U, \ \text{ if } \lambda \text{ is large enough}$$

Deduce, by Maximum Principle that

$$\|Du\|_{L^\infty(U)} \leq C\|Du\|_{L^\infty(\partial\Omega)} + C\|u\|_{L^\infty(\partial\Omega)}$$

6. Let u be a smooth function satisfying

$$-\Delta u + V(x)u = f(x), |u| \leq 1, \ \text{ in } R^n$$

where

$$|f(x)| \leq Ce^{-|x|}$$

and

$$V(x) \geq 2 \ \text{ for } |x| > 1$$

Deduce from maximum principle that u actually decays

$$|u(x)| \leq Ce^{-|x|}$$