Solutions to Midterm MATH400-101, 2016-2017

1. The equation of characteristic is

$$\frac{dx}{y} = \frac{dy}{2xy}$$

so $2x\,dx = dy$

$$\Rightarrow \quad x^2 - y = \xi$$

Let $\begin{cases} x' = x \\ y' = x^2 - y \end{cases}$, $u(x,y) = U(x', y')$

Then

$$y U_{x'} = 2x U$$

$$U_{x'} = \frac{2x}{y} U = \frac{2x'}{x'^2 - y'} U$$

$$\Rightarrow \ln U = c(y') + \ln(x'^2 - y')$$

$$U = f(y')(x'^2 - y')$$

$$u = f(x^2 - y)\, y$$

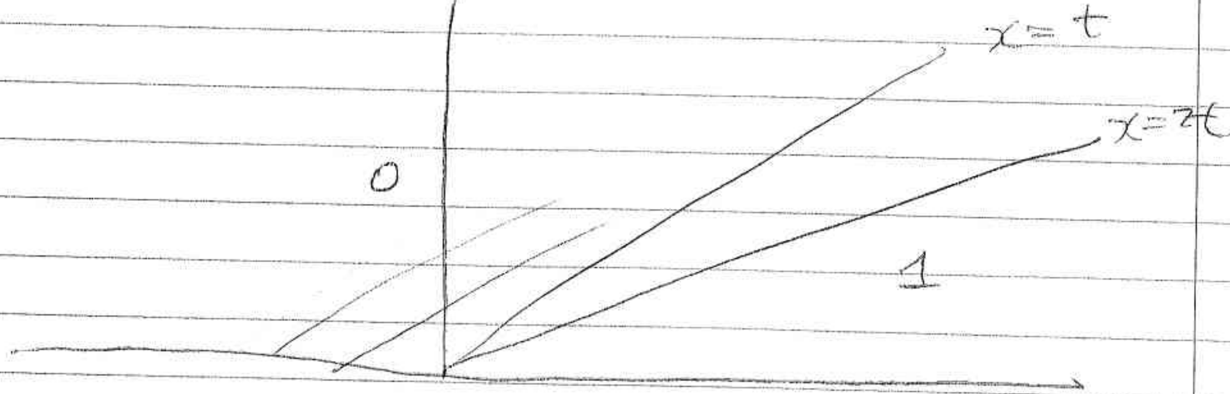
2. $c(u) = 2 - u$, $Q(u) = 2u - \frac{u^2}{2}$, $\frac{Q(u^+) - Q(u^-)}{u^+ - u^-} = 2 - \frac{1}{2}(u^+ + u^-)$

$$u(x,0) = \begin{cases} 0, & -\infty < x < 0 \\ 1, & 0 < x < +\infty \end{cases}$$

$x = c(f(\xi))t + \xi \quad \Rightarrow$

$\xi < 0$, $x = (2-0)t + \xi = 2t + \xi$

$0 < \xi$, $x = (2-1)t + \xi = t + \xi$



Shock: $\frac{ds}{dt} = 2 - \frac{1}{2}(1+0) = \frac{3}{2}$

$s(0)=0$

$s(t) = \frac{3}{2}t$

So $u(x,t) = \begin{cases} 0, & -\infty < x < \frac{3}{2}t \\ 1 & \frac{3}{2}t < x < +\infty \end{cases}$

3. $\partial_t^2 = \partial_t \partial_x$ $\quad \partial_t(\partial_t - \partial_x) = 0$ $\quad \partial_t = \partial_\xi$, $\partial_x = \partial_\xi - \partial_\eta$

$\partial_t = \partial_\xi$ $\quad \partial_x = \partial_\xi - \partial_\eta$

$\partial_t - \partial_x = \partial_\eta$ $\quad \partial_t = \partial_\xi$

$\cancel{\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}t\\ x\end{pmatrix}}$ $\quad \cancel{\xi = t - x}$ $\quad \eta = x$

$\begin{pmatrix}\partial t\\ \partial x\end{pmatrix} = \begin{pmatrix}\partial_\xi\\ \partial_\xi - \partial_\eta\end{pmatrix} = \begin{pmatrix}1 & 0\\ +1 & -1\end{pmatrix}\begin{pmatrix}\partial_\xi\\ \partial_\eta\end{pmatrix}$

$\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \begin{pmatrix}1 & 1\\ 0 & -1\end{pmatrix}\begin{pmatrix}t\\ x\end{pmatrix}$

$$u = f(x+t) + g(-x)$$
$$= F(x+t) + G(x)$$

$$u(x,0)=0 \Rightarrow F(x)+G(x)=0, \quad x>0$$

$$u_t(x,0)=0 \Rightarrow F'(x)=0 \Rightarrow F(x)=F(0)$$
$$G(x) = -F(0)$$

$$u(0,t)=0 \Rightarrow F(t)+G(0)=t$$

$$\Rightarrow F(t) = t - G(0) \quad t>0.$$
$$F(x) = x - G(0), \quad G(x) = -x + G(0)$$

$$u(x,t) = \cancel{x+t - G(0) + F(0) = x+t =}$$

$$= x+t - G(0) + (-x + G(0))$$

$$= t$$

4. $$u(x,t) = \frac{1}{2}[0+0] + \frac{1}{2c}\int 0 + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} \cos y \, dy\, ds$$

$$= \frac{1}{2c}\int_0^t \sin y \Big|_{x-c(t-s)}^{x+c(t-s)} ds$$

$$= \frac{2}{2c}\left(\int_0^t \sin x \cos c(t-s)\, ds\right)$$

$$= \frac{1}{c}\sin x \; \frac{1}{c}\sin c(s-t) \Big|_0^t = -\frac{\sin x}{c^2}\sin ct$$

5.

$$u(x,t) = \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{(x-y)^2}{4kt}} e^{-y^2} dy$$

$$= \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{(x-y)^2}{t} - y^2} dy$$

$$\frac{(x-y)^2}{t} + y^2 = \left(\frac{1}{t}+1\right) y^2 - \frac{2x}{t} y + \frac{x^2}{t}$$

$$= \left(\frac{1}{t}+1\right)\left(y^2 - \frac{2x}{t+1} y + \left(\frac{2x}{t+1}\right)^2\right) - \frac{t+1}{t}\left(\frac{2x}{t+1}\right)^2 + \frac{x^2}{t}$$

$$= \left(\frac{1}{t}+1\right)\left(y - \frac{2x}{t+1}\right)^2 + \frac{x^2}{t} - \frac{4x^2}{t(t+1)}$$

$$u(x,t) = \frac{1}{\sqrt{2\pi t}} \int e^{-\left(1+\frac{1}{t}\right)\left(y - \frac{2x}{t+1}\right)^2} dy \; e^{-\frac{x^2}{t} + \frac{4x^2}{t(t+1)}}$$

$$y = \left(\frac{2x}{t+1}\right) + \frac{1}{\sqrt{1+\frac{1}{t}}} p.$$

$$u = \frac{1}{\sqrt{t}} \cdot \frac{1}{\sqrt{1+\frac{1}{t}}} \; e^{-\frac{x^2}{t} + \frac{4x^2}{t(t+1)}}$$

$$= \frac{1}{\sqrt{t+1}} e^{-\frac{x^2}{t+1}}$$