Be sure this exam has 9 pages including the cover

The University of British Columbia

MATH 400, Sections 101

Midterm II –November 16, 2018

Name ______________________ Signature ______________________

Student Number ______________

This exam consists of 4 questions. No notes. Write your answer in the blank page provided.

Problem	max score	score
1.	30	
2.	20	
3.	25	
4.	25	
total	100	

1. Each candidate should be prepared to produce his library/AMS card upon request.

2. Read and observe the following rules:

No candidate shall be permitted to enter the examination room after the expiration of one half hour, or to leave during the first half hour of the examination.

Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.

CAUTION - Candidates guilty of any of the following or similar practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.

(a) Making use of any books, papers or memoranda, other than those authorized by the examiners.

(b) Speaking or communicating with other candidates.

(c) Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness shall not be received.

3. Smoking is not permitted during examinations.

(30 points) 1. This problem contains two parts

(15 points) (a) Solve the following heat equation

$$4u_t = u_{xx} + 4u, -\infty < x < +\infty, t > 0$$

$$u(x,0) = e^{-x^2}$$

Hint: let $u(x,t) = e^t v(x,t)$.

(15 points) (b) Solve the following wave equation

$$u_{tt} = u_{xx} + 1, 0 < x < +\infty, t > 0$$

$$u(x,0) = 1, u_t(x,0) = 0, x > 0$$

$$u(0,t) = 0, t > 0$$

1 (a) Let $u(x,t) = e^t v(x,t)$. Then $v(x,t)$ satisfies

$$\begin{cases} 4V_t = V_{xx} \\ v(x,0) = e^{-x^2} \end{cases}$$

+3

With $k = \frac{1}{4}$, we have

$$v(x,t) = \frac{1}{\sqrt{\pi t}} \int_{-\infty}^{+\infty} e^{-\frac{(x-y)^2}{t} - y^2} dy$$

+4

$$\frac{(x-y)^2}{t} + y^2 = \left(\frac{1}{t}+1\right)y^2 - \frac{2x}{t}y + \frac{x^2}{t}$$

$$= \left(\frac{1}{t}+1\right)\left(y - \frac{2}{t}\cdot\frac{x}{\frac{1}{t}+1}\right)^2 + \frac{x^2}{t} - \frac{\left(\frac{x}{t}\right)^2}{\frac{1}{t}+1}$$

$$= \left(\frac{1}{t}+1\right)\left(y - \frac{2x}{t+1}\right)^2 + \frac{x^2}{t}\left(1 - \frac{t}{t+1}\right)$$

$$= \left(\frac{1}{t}+1\right)\left(y - \frac{2x}{t+1}\right)^2 + \frac{x^2}{t+1}$$

+8

$$v = \frac{1}{\sqrt{\pi t}} \int_{-\infty}^{+\infty} e^{-\left(\frac{1}{t}+1\right)\left(y - \frac{2x}{t+1}\right)^2} dy \; e^{-\frac{x^2}{t+1}}$$

$$= \frac{1}{\sqrt{\pi t}} \sqrt{\frac{t}{t+1}} \cdot \int_{-\infty}^{+\infty} e^{-p^2} dy \; e^{-\frac{x^2}{t+1}} \qquad \left(y = \frac{2x}{t+1} + \frac{p}{\sqrt{\frac{1}{t}+1}}\right)$$

$$= \frac{1}{\sqrt{\pi(t+1)}} e^{-\frac{x^2}{t+1}}$$

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So $u(x,t) = \frac{1}{\sqrt{t+1}} e^{-\frac{x^2}{t+1}} e^{t}$

Method 3: $u(x,t) = v(x,t) + \frac{t^2}{2}$
then use general sol'n

(b). There are ~~three~~ ways to solve the problem:

Method 1: Extension

$$f_{ext} = \begin{cases} 1, & x>0 \\ -1, & x<0 \end{cases}, \quad \phi_{ext} = \begin{cases} 1, & x>0 \\ -1, & x<0 \end{cases}$$ (+3)

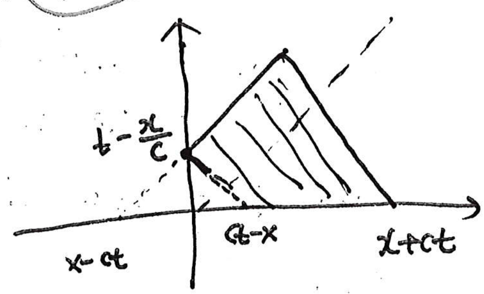
$$u(x,t) = \frac{1}{2}\left[\phi_{ext}(x-ct) + \phi_{ext}(x+ct)\right] + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f_{ext}(y,s)\,dy\,ds$$ (+3)

when $x>ct$, we have

$$u = 1 + \frac{1}{2c}\int_0^t 2c(t-s)\,ds = 1 + \frac{t^2}{2}$$ (+3)

when $x<ct$, we have

$$u = 0 + \frac{1}{2c}\int_0^t \int_{x-c(t-s)}^{x+c(t-s)} f_{ext}(y,s)\,dy\,ds$$



$= \frac{1}{2c}$ area of shaded region (+6)

$$= \frac{1}{2c}\left(ct^2 - (ct-x)\cdot\left(t-\frac{x}{c}\right)\right)$$

$$= \frac{1}{2c}\left(ct^2 - \frac{(ct-x)^2}{c}\right) = \frac{1}{2c^2}\left(2ctx - x^2\right) = \frac{tx}{c} - \frac{x^2}{c^2}$$

So $u(x,t) = \begin{cases} 1+\frac{t^2}{2}, & x>ct \\ \frac{tx}{c} - \frac{x^2}{2c^2}, & x<ct \end{cases}$

Method 2: Let $u(x,t) = -\frac{x^2}{2} + v(x,t)$. Then

$$\begin{cases} v_{tt} = v_{xx} \\ v(x,0) = 1+\frac{x^2}{2}, \quad v_t(x,0) = 0 \\ v(0,t) = 0 \end{cases} \Rightarrow v = \frac{1}{2}\left[\phi_{ext}(x-ct) + \phi_{ext}(x+ct)\right] = \begin{cases} 1+\frac{t^2}{2}+\frac{x^2}{2}, & x>ct \\ 2tx & x<ct \end{cases}$$

(20 points) 2. This problem contains two parts.

(10 points) (a) Prove that the diffusion equation

$$u_t = ku_{xx} + f(x,t), 0 < x < +\infty, t > 0$$

$$u(x,0) = \phi(x), 0 < x < +\infty$$

$$u(0,t) = h(t), t > 0$$

is well-posed.

Hint: let $u(x,t) = v(x,t) + h(t)$ and use the method of reflection to find the solution formula.

(10 points) (b) Discuss one difference between wave equation and heat equation. Justify your answer.

2. (a). By the method of extension, $\Rightarrow$

let $u(x,t) = v(x,t) + h(t)$.

Then v satisfies

$$\begin{cases} v_t = kv_{xx} + f(x,t) - h'(t) \\ v(x,0) = \phi(x) - h(0) \\ v(0,t) = 0 \end{cases}$$

Let $\tilde{f} = f(x,t) - h'(t)$, $\tilde{\phi} = \phi(x) - h(0)$

Then

$$v(x,t) = \int_{-\infty}^{+\infty} S(x-y)\, \tilde{\phi}_{ext}(y)\,dy + \int_0^t \int_{-\infty}^{+\infty} S(x-y, t-s)\, \tilde{f}_{ext}(y,s)\,dy\,ds$$

where $\tilde{\phi}_{ext} = \begin{cases} \tilde{\phi}(x), & x > 0 \\ -\tilde{\phi}(-x), & x < 0 \end{cases}$; $\tilde{f}_{ext} = \begin{cases} \tilde{f}(x,t), & x > 0 \\ -\tilde{f}(-x,t), & x < 0 \end{cases}$ (+3)

Well-posedness: 1) Existence: By the formula (+2)

2) Uniqueness: By the formula (+2)

3) Stability:

$$|v(x,t)| \le \max|\tilde{\phi}| + t\max|\tilde{f}|$$
$$\le \max(|\phi| + |h|) + t\max(|f| + |h'|)$$ (+3)

This proves stability.

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(2). Consider the energy for wave equation

$$u_{tt} = c^2 u_{xx}$$

$$E(t) = \frac{1}{2}\int u_t^2 + \frac{c^2}{2}\int u_x^2$$

+5

Then $\frac{dE}{dt} = \int u_t u_{tt} + c^2 \int u_x u_{tx}$

$$= \int u_t c^2 u_{xx} + c^2 \int u_x u_{tx}$$

$$= c^2 \int (u_t u_x)_x = 0$$

So $E(t) = E(0)$.

On the other hand, for the energy of heat equation

+5

$$u_t = k u_{xx}$$

$$E(t) = \frac{1}{2}\int u^2(x,t)\,dx$$

$$\frac{dE}{dt} = \int u\, u_t = \int u(k u_{xx}) = -k\int u_x^2 \leq 0.$$

$$E(t) \leq E(0).$$

(25 points) 3. Consider the following heat equation

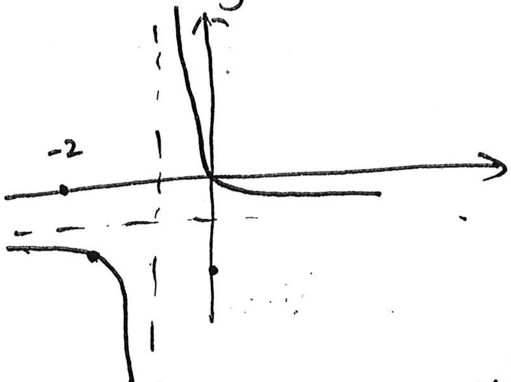
$$u_t = u_{xx} \quad 0 < x < 1, \ \ t > 0$$
$$u(x, 0) = \phi(x), \ \ 0 < x < 1$$
$$u_x(0, t) + 2u(0, t) = 0, \qquad u_x(1, t) - 2u(1, t) = 0, \ t > 0$$

(i) (20) Use the method of separation of variables to find the general solution. (You should give the equations for negative, zero or positive eigenvalues, and general solution with coefficients in terms of the eigenfunctions and ϕ.)

(ii) (5) What can you say about the asymptotic behavior of u as $t \to +\infty$? Justify your answer.

3. Here $k=1$, $l=1$, $a_0=-2$, $a_l=2$ (+3)

(i). In the region



(a_0, a_l) in the region (IV): (+3)

So there is one negative eigenvalue

$\lambda_{-1} = -\gamma_{-1}^2 < 0$, $\tanh(\gamma_{-1}) = \dfrac{4\gamma_{-1}}{\gamma_{-1}^2+4}$ (3)

$X_{-1} = \cosh \gamma_{-1} x - \dfrac{2}{\gamma_{-1}} \sinh \gamma_{-1} x$

there is a zero eigenvalue (3)

$\lambda_0 = 0$, $X_0 = 1 - 2x$

there are infinitely many positive eigenvalue (2)

$\lambda_n = \beta_n^2 > 0$, $\tan \beta_n = \dfrac{-4}{\beta^2 - 4}$

$X_n = \cos \beta_n x - \dfrac{2}{\beta_n} \sin \beta_n x$

$$u(x,t) = e^{+\gamma_1^2 t} a_{-1} X_{-1}(x) + a_0 X_0(x) + \sum_{n=1}^{+\infty} a_n e^{-\beta_n^2 t} X_n(x)$$ (+5)

$$a_n = \frac{\int_0^1 \phi(x) X_n(x)\, dx}{\int_0^1 X_n^2(x)\, dx}, \quad n = -1, 0, 1, 2, \dots$$

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(ii) If $a_{-1} \neq 0$, then $|u(x,t)| \to +\infty$ as $t \to +\infty$.

If $a_{-1} = 0$, then u is bounded as $t \to +\infty$.

If $a_{-1} = a_0 = 0$, then $u(x,t) \to 0$ as $t \to +\infty$.

+5

(25 points) 4. Consider the following eigenvalue problem

$$x^2X'' - xX' + \lambda X = 0, 1 < x < e$$
$$X(1) = 0, X(e) = 0.$$

(5 points) (a) Write the eigenvalue problem in standard Sturm-Liouville form $(pX')' - qX + \lambda wX = 0$.

(10 points) (b) Solve the eigenvalue problem. (Hint: For Euler's type equation $aX'' + bX' + cX = 0$ we let $X = x^r$ where r satisfies the characteristic root equation $ar(r-1) + br + c = 0$. If $r_1 \neq r_2$ then $X = c_1x^{r_1} + c_2x^{r_2}$; if $r_1 = r_2 = r$ then $X = c_1x^r + c_2x^r \log x$; if $r = \lambda + i\mu$ is a complex root, then $X = c_1x^\lambda \cos(\mu \log x) + c_2x^\lambda \sin(\mu \log x)$.)

(10 points) (c) Use the method of separation of variables to find the solution to the following wave equation

$$u_{tt} = x^2u_{xx} - xu_x, 1 < x < e$$
$$u(1,t) = 0, u(e,t) = 0$$
$$u(x,0) = 0, u_t(x,0) = x, 1 < x < e$$

You may use results in part (b).

(a). $X'' - \frac{1}{x}X' + \frac{\lambda}{x^2}X = 0$

$p = \mu$, $p' = -\frac{1}{x}\mu$, $w = \frac{1}{x^2}\mu$

$\frac{p'}{p} = -\frac{1}{x} \Rightarrow p = \frac{1}{x}$, $\mu = \frac{1}{x}$, $w = x^{-3}$

$(x^{-1}X')' + \lambda x^{-3}X = 0$

(+5)

(b). $x^2X'' - xX' + \lambda X = 0 \Rightarrow r(r-1) - r + \lambda = 0$

$r^2 - 2r + \lambda = 0$, $r = 1 \pm \sqrt{4-\lambda}$

(+2)

Case 1. $\lambda < 1 \Rightarrow r_1 \neq r_2$

$X = C_1x^{r_1} + C_2x^{r_2} \Rightarrow X(1) = C_1 + C_2 = 0$, $X(e) = C_1e^{r_1} + C_2e^{r_2} = 0$ $\Big\} \Rightarrow C_1 = C_2 = 0$

(+1)

Case 2 $\lambda = 1 \Rightarrow r_1 = r_2 = 1$

$X = C_1x + C_2x\ln x \Rightarrow X(1) = 0 \Rightarrow C_1 = 0$

$X(e) = 0 \Rightarrow C_2 = 0$

(+1)

Case 3 $\lambda > 1$. $X = C_1x\cos(\sqrt{\lambda-1}\ln x) + C_2x\sin(\sqrt{\lambda-1}\ln x)$

$X(1) = 0 \Rightarrow C_1 = 0$, $X(e) = 0 \Rightarrow C_2e\sin(\sqrt{\lambda-1}) = 0$

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$$\sqrt{\lambda-1} = n\pi, \quad n=1,2,\cdots \qquad +6$$

$$\lambda_n = 1+(n\pi)^2, \quad n=1,2,\cdots$$

$$X_n = x\sin(n\pi\ln x)$$

(c). $T'' + \lambda_n T = 0 \Rightarrow T = c_1\cos\sqrt{\lambda_n}\,t + c_2\sin(\sqrt{\lambda_n}\,t)$ +2

$$u = \sum_{n=1}^{+\infty} (x\sin(n\pi\ln x))(a_n\cos\sqrt{\lambda_n}\,t + b_n\sin\sqrt{\lambda_n}\,t) \qquad +3$$

+2

$u(x,0)=0 \Rightarrow a_n = 0$

$u_t(x,0)=x \Rightarrow x = \sum_{n=1}^{+\infty} \sqrt{\lambda_n}\, b_n\, x\sin(n\pi\ln x)$

$$\sqrt{\lambda_n}\, b_n = \frac{\int_1^e x^{-3}\, x\cdot x\sin(n\pi\ln x)\,dx}{\int_1^e x^{-3}\,(x\sin(n\pi\ln x))^2\,dx} \qquad +3$$

$$\int_1^e x^{-1}\sin^2(n\pi\ln x)\,dx \overset{t=\ln x}{=} \int_0^1 \sin^2(n\pi t) = \frac{1}{2}$$

(It is possible to compute b_n:

$$\int_1^e x^{-1}\sin(n\pi\ln x)\,dx = \int_0^1 \sin(n\pi t)\,dt = \frac{1}{n\pi}(1-(-1)^n)\,)$$

extra credit if you got this.