

Be sure this exam has 6 pages including the cover

The University of British Columbia

MATH 400, Section 101

Midterm Exam I – October 12 2018

Name _____ Signature _____

Student Number _____

This exam consists of 4 questions. No notes. Simple numerics calculators are allowed. Write your answer in the blank page provided.

Problem	max score	score
1.	20	
2.	20	
3.	40	
4.	20	
total	100	

1. Each candidate should be prepared to produce his library/AMS card upon request.
2. Read and observe the following rules:
No candidate shall be permitted to enter the examination room after the expiration of one half hour, or to leave during the first half hour of the examination.
Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
CAUTION - Candidates guilty of any of the following or similar practices shall be immediately dismissed from the examination and shall be liable to disciplinary action.
 - (a) Making use of any books, papers or memoranda, other than those authorized by the examiners.
 - (b) Speaking or communicating with other candidates.
 - (c) Purposely exposing written papers to the view of other candidates. The plea of accident or forgetfulness shall not be received.
3. Smoking is not permitted during examinations.

(20 points) 1. Solve the following first order PDE:

$$3u_x - 2u_y + u = x$$

$$u(x, x) = x, -\infty < x < +\infty$$

Solution: Use method of characteristics.

$$\begin{cases} \frac{dx}{ds} = 3, & x(0) = 3 \\ \frac{dy}{ds} = -2, & y(0) = 3 \\ \frac{du}{ds} = -u + x, & u(0) = 3 \end{cases}$$

So $x = 3s + 3, y = -2s + 3$

$$\frac{du}{ds} = -u + 3s + 3 \Rightarrow u_p = As + B \Rightarrow A = -As - B + 3s + 3$$

$$A = 3, A + B = 3, B = 3 - 3$$

$$u = 3s + 3 - 3 + ce^{-s}$$

$$u(0) = 3 \Rightarrow 3 - 3 + c = 3 \Rightarrow c = 3$$

So $u = 3(s-1) + 3 + 3e^{-s}$

$$s = \frac{1}{5}(x-y), \quad 3 = \frac{2}{5}x + \frac{3}{5}y$$

$$u = 3\left(\frac{1}{5}(x-y) - 1\right) + \frac{2}{5}x + \frac{3}{5}y + 3e^{-\frac{1}{5}x + \frac{1}{5}y}$$

(20 points) 2. Find the general solutions to the following first order PDE:

$$2u_x + 2xyu_y = u$$

Solution: $\frac{dx}{2} = \frac{dy}{2xy} \Rightarrow xdx = \frac{1}{y} dy$

$$\frac{x^2}{2} = \ln y + c \Rightarrow e^{\frac{x^2}{2}} = cy \Rightarrow z = ye^{-\frac{x^2}{2}}$$

Let $\begin{cases} x' = x \\ y' = z \end{cases}, \quad u = U$

Then $2u_x = 2U_{x'} = U \Rightarrow U = f(z)e^{\frac{x'}{2}}$

$$u = f(ye^{-\frac{x^2}{2}}) e^{\frac{x}{2}}$$

(40 points) 3. Consider the following first order PDE:

$$u_t + u^2 u_x = 0$$

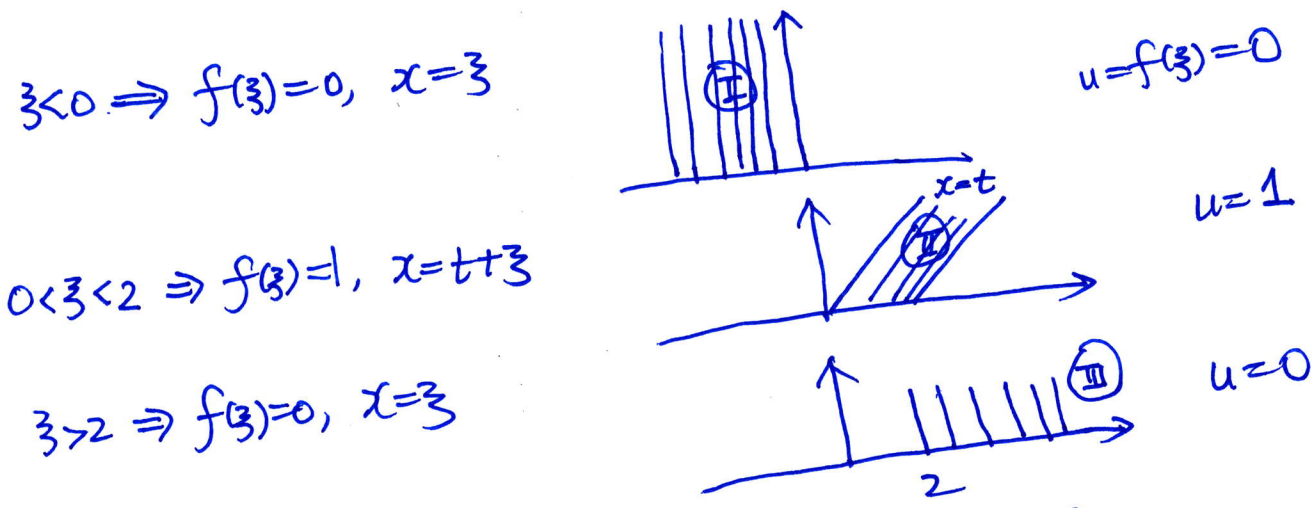
$$u(x, 0) = \begin{cases} 0, & x < 0, \\ 1, & 0 < x < 2, \\ 0, & x > 2 \end{cases}$$

- (10 points) (a) Find the expansion fan solutions.
- (20 points) (b) Find the solution before the shock curve hits the expansion fan.
- (10 points) (c) Find the solution after the shock curve hits the expansion fan.

Solutions: (a) $U^2(\lambda) = \lambda \Rightarrow U = \pm \sqrt{\lambda}$ "+" only \rightarrow 8 pts

$$u = \pm \sqrt{\frac{x}{t}}$$

(b) $x = f^2(\xi) t + \xi$, $f(\xi) = \begin{cases} 0, & \xi < 0 \\ 1, & 0 < \xi < 2 \\ 0, & \xi > 2 \end{cases}$, $u = f(\xi)$



Between I & II, insert expansion fan, $u = \sqrt{\frac{x}{t}}$
 Between II & III, \exists a shock curve $\frac{ds}{dt} = \frac{Q^+ - Q^-}{u^+ - u^-} = \frac{\frac{1}{3}u^{+3} - \frac{1}{3}u^{-3}}{u^+ - u^-}$
 $= \frac{0 - \frac{1}{3}}{0 - 1} = \frac{1}{3}$, $s(0) = 2$

$s = \frac{1}{3}t + 2$
 shock curve hits the expansion fan.
 $\left. \begin{matrix} x=t \\ x = \frac{1}{3}t + 2 \end{matrix} \right\} \Rightarrow t=3, x=3$

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$$u(x,t) = \begin{cases} 0, & x < 0 \\ \sqrt{\frac{x}{t}}, & 0 < x < t \\ 1, & t < x < \frac{1}{3}t + 2 \\ 0, & x > \frac{1}{3}t + 2 \end{cases}$$

(c) when the shock curve hits the expansion fan

$$\left\{ \begin{aligned} \frac{ds}{dt} &= \frac{\frac{1}{3}(u^+)^3 - \frac{1}{3}(u^-)^3}{u^+ - u^-} = \frac{0 - \frac{1}{3}u^{+3}}{0 - u^+} = \frac{1}{3}u^{+2} = \frac{1}{3}\frac{x}{t} = \frac{1}{3}\frac{s}{t} \\ s(3) &= 3 \end{aligned} \right.$$

$$\frac{ds}{s} = \frac{1}{3t} dt \Rightarrow \ln s = \frac{1}{3} \ln t + C$$

$$\Rightarrow s = C t^{\frac{1}{3}}$$

$$3 = s(3) = C \cdot 3^{\frac{1}{3}} \Rightarrow C = 3^{\frac{2}{3}}$$

$$s = 3^{\frac{2}{3}} t^{\frac{1}{3}}$$

$$u(x,t) = \begin{cases} 0, & x < 0 \\ \sqrt{\frac{x}{t}}, & 0 < x < 3^{\frac{2}{3}} t^{\frac{1}{3}} \\ 0, & x > 3^{\frac{2}{3}} t^{\frac{1}{3}} \end{cases}, \quad t > 3$$

(20 points) 4. Consider the following second order PDE:

$$u_{xx} + 4u_{xy} - 5u_{yy} = 0$$

(10 points) (a) Find the linear transformation so that it becomes standard hyperbolic type $u_{\xi\xi} + u_{\eta\eta} = 0$.

(10 points) (b) Find another linear transformation so that it becomes $u_{\xi\eta} = 0$. Use it to find the general solutions to $u_{xx} + 4u_{xy} - 5u_{yy} = 0$.

$$\begin{aligned} \text{(a). } & \partial_x^2 + 4\partial_x\partial_y - 5\partial_y^2 \\ & = (\partial_x + 2\partial_y)^2 - (3\partial_y)^2 \\ & \partial_z = \partial_x + 2\partial_y \Rightarrow \partial_x = \partial_z - \frac{2}{3}\partial_\eta \\ & \partial_\eta = 3\partial_y \Rightarrow \partial_y = \frac{1}{3}\partial_\eta \end{aligned}$$

$$\begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} = \begin{pmatrix} 1 & -\frac{2}{3} \\ 0 & \frac{1}{3} \end{pmatrix} \begin{pmatrix} \partial_z \\ \partial_\eta \end{pmatrix}$$

$$\text{Thus } \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\frac{2}{3} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} z \\ \eta \end{pmatrix}$$

$$\text{(b). } \partial_x^2 + 4\partial_x\partial_y - 5\partial_y^2 = (\partial_x + 5\partial_y)(\partial_x - \partial_y)$$

$$\begin{aligned} \partial_z &= \partial_x + 5\partial_y \\ \partial_\eta &= \partial_x - \partial_y \end{aligned} \Rightarrow \begin{aligned} \partial_x &= \frac{1}{6}\partial_z + \frac{5}{6}\partial_\eta \\ \partial_y &= \frac{1}{6}\partial_z - \frac{1}{6}\partial_\eta \end{aligned}$$

$$\begin{pmatrix} \partial_x \\ \partial_y \end{pmatrix} = \begin{pmatrix} \frac{1}{6} & \frac{5}{6} \\ \frac{1}{6} & -\frac{1}{6} \end{pmatrix} \begin{pmatrix} \partial_z \\ \partial_\eta \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{6} & \frac{1}{6} \\ \frac{5}{6} & -\frac{1}{6} \end{pmatrix} \begin{pmatrix} z \\ \eta \end{pmatrix}$$

$$\xi = \frac{1}{6}(x+y), \quad \eta = \frac{1}{6}(5x-y)$$

so general sol'n becomes

$$u = f(\xi) + g(\eta) = f\left(\frac{1}{6}(x+y)\right) + g\left(\frac{1}{6}(5x-y)\right)$$