Be sure this exam has 6 pages including the cover

The University of British Columbia

MATH 400, Section 101

Midterm Exam I – October 12 2018

Name ______________________ Signature ______________________

Student Number ______________________

This exam consists of 4 questions. No notes. Simple numerics calculators are allowed. Write your answer in the blank page provided.

Problem	max score	score
1.	20	
2.	20	
3.	40	
4.	20	
total	100	

(20 points) 1. Solve the following first order PDE:

$$3u_x - 2u_y + u = x$$

$$u(x,x) = x, -\infty < x < +\infty$$

Solution: Use method of characteristics.

$$\begin{cases} \dfrac{dx}{ds} = 3, & x(0) = \xi \\ \dfrac{dy}{ds} = -2, & y(0) = \xi \\ \dfrac{du}{ds} = -u + x, & x(0) = \xi \end{cases}$$

So $x = 3s + \xi$, $y = -2s + \xi$

$$\frac{du}{ds} = -u + 3s + \xi \Rightarrow u_p = As + B \Rightarrow A = -As - B + 3s + \xi$$

$$A = 3, \quad A + B = \xi, \quad B = \xi - 3$$

$$u = 3s + \xi - 3 + ce^{-s}$$

$$u(0) = \xi \Rightarrow \xi - 3 + c = \xi \Rightarrow c = 3$$

So $u = 3(s-1) + \xi + 3e^{-s}$

$$s = \frac{1}{5}(x-y), \quad \xi = \frac{2}{5}x + \frac{3}{5}y$$

$$u = 3\left(\frac{1}{5}(x-y) - 1\right) + \frac{2}{5}x + \frac{3}{5}y + 3e^{-\frac{1}{5}x + \frac{1}{5}y}$$

(20 points) 2. Find the general solutions to the following first order PDE:

$$2u_x + 2xyu_y = u$$

Solution: $\frac{dx}{2} = \frac{dy}{2xy} \Rightarrow x\,dx = \frac{1}{y}dy$

$\frac{x^2}{2} = \ln y + c \Rightarrow e^{\frac{x^2}{2}} = cy \Rightarrow \xi = ye^{-\frac{x^2}{2}}$

Let $\begin{cases} x' = x \\ y' = \xi \end{cases}$, $u = U$

Then $2u_x = 2U_{x'} = U \Rightarrow U = f(\xi)e^{\frac{x'}{2}}$

$u = f(ye^{-\frac{x^2}{2}})\,e^{\frac{x}{2}}$

(40 points) 3. Consider the following first order PDE:

$$u_t + u^2 u_x = 0$$

$$u(x,0) = \begin{cases} 0, x < 0, \\ 1, 0 < x < 2, \\ 0, x > 2 \end{cases}$$

(10 points) (a) Find the expansion fan solutions.

(20 points) (b) Find the solution before the shock curve hits the expansion fan.

(10 points) (c) Find the solution after the shock curve hits the expansion fan.

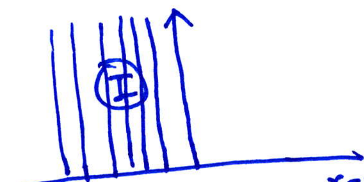
Solutions: (a) $U^2(\lambda) = \lambda \Rightarrow U = \pm\sqrt{\lambda}$

$\boxed{\text{"+" only} \rightarrow 8 \text{ pts}}$

$u = \pm\sqrt{\frac{x}{t}}$

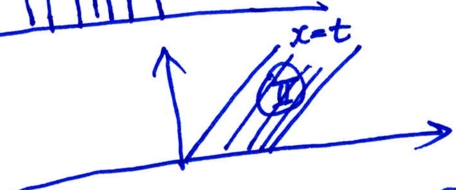
(b) $x = f^2(\xi)t + \xi$, $f(\xi) = \begin{cases} 0, & \xi < 0 \\ 1, & 0<\xi<2 \\ 0, & \xi > 2 \end{cases}$, $u = f(\xi)$

$\xi < 0 \Rightarrow f(\xi) = 0$, $x = \xi$



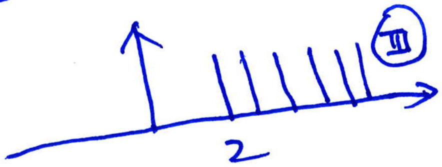
$u = f(\xi) = 0$

$0<\xi<2 \Rightarrow f(\xi) = 1$, $x = t+\xi$



$u = 1$

$\xi > 2 \Rightarrow f(\xi) = 0$, $x = \xi$



$u = 0$

Between I & II, insert expansion fan, $u = \sqrt{\frac{x}{t}}$

Between II & III, $\exists$ a shock curve $\frac{ds}{dt} = \frac{Q^+ - Q^-}{u^+ - u^-} = \frac{\frac{1}{3}u^{+3} - \frac{1}{3}u^{-3}}{u^+ - u^-}$

$= \frac{0 - \frac{1}{3}}{0 - 1} = \frac{1}{3}$, $s(0) = 2$

$s = \frac{1}{3}t + 2$

Shock curve hits the expansion fan: $\left.\begin{array}{l} x = t \\ x = \frac{1}{3}t + 2 \end{array}\right\} \Rightarrow t = 3, x = 3$

(continuing blank page)

$$u(x,t)=\begin{cases} 0, & x<0 \\ \sqrt{\frac{x}{t}}, & 0<x<t \\ 1, & t<x<\frac{1}{3}t+2 \\ 0, & x>\frac{1}{3}t+2 \end{cases}$$

(c) when the shock curve hits the expansion fan

$$\begin{cases} \frac{ds}{dt} = \frac{\frac{1}{3}(u^+)^3-\frac{1}{3}(u^-)^3}{u^+-u^-} = \frac{0-\frac{1}{3}u^{+3}}{0-u^+} = \frac{1}{3}u^{+2} = \frac{1}{3}\frac{x}{t} = \frac{1}{3}\frac{s}{t} \\ s(3)=3 \end{cases}$$

$$\frac{ds}{s} = \frac{1}{3t}dt \Rightarrow \ln s = \frac{1}{3}\ln t + C$$

$$\Rightarrow s = C t^{\frac{1}{3}}$$

$$3=s(3)=C\cdot 3^{\frac{1}{3}} \Rightarrow C=3^{\frac{2}{3}}$$

$$s=3^{\frac{2}{3}}t^{\frac{1}{3}}$$

$$u(x,t)=\begin{cases} 0, & x<0 \\ \sqrt{\frac{x}{t}}, & 0<x<3^{\frac{2}{3}}t^{\frac{1}{3}} \\ 0, & x>3^{\frac{2}{3}}t^{\frac{1}{3}} \end{cases}, \quad t>3$$

(20 points) 4. Consider the following second order PDE:

$$u_{xx} + 4u_{xy} - 5u_{yy} = 0$$

(10 points) (a) Find the linear transformation so that it becomes standard hyperbolic type $u_{\xi\xi} + u_{\eta\eta} = 0$.

(10 points) (b) Find another linear transformation so that it becomes $u_{\xi\eta} = 0$. Use it to find the general solutions to $u_{xx} + 4u_{xy} - 5u_{yy} = 0$.

4

(a). $\partial_x^2 + 4\partial_x\partial_y - 5\partial_y^2$

$= (\partial_x + 2\partial_y)^2 - (3\partial_y)^2$

$\partial_\xi = \partial_x + 2\partial_y$
$\partial_\eta = 3\partial_y$ $\Rightarrow$ $\partial_x = \partial_\xi - \frac{2}{3}\partial_\eta$, $\partial_y = \frac{1}{3}\partial_\eta$

$$\begin{pmatrix}\partial_x\\ \partial_y\end{pmatrix} = \begin{pmatrix}1 & -\frac{2}{3}\\ 0 & \frac{1}{3}\end{pmatrix}\begin{pmatrix}\partial_\xi\\ \partial_\eta\end{pmatrix}$$

Thus
$$\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \begin{pmatrix}1 & 0\\ -\frac{2}{3} & \frac{1}{3}\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

(b). $\partial_x^2 + 4\partial_x\partial_y - 5\partial_y^2 = (\partial_x + 5\partial_y)(\partial_x - \partial_y)$

$\partial_\xi = \partial_x + 5\partial_y$
$\partial_\eta = \partial_x - \partial_y$ $\Big\} \Rightarrow$ $\partial_x = \frac{1}{6}\partial_\xi + \frac{5}{6}\partial_\eta$, $\partial_y = \frac{1}{6}\partial_\xi - \frac{1}{6}\partial_\eta$

$$\begin{pmatrix}\partial_x\\ \partial_y\end{pmatrix} = \begin{pmatrix}\frac{1}{6} & \frac{5}{6}\\ \frac{1}{6} & -\frac{1}{6}\end{pmatrix}\begin{pmatrix}\partial_\xi\\ \partial_\eta\end{pmatrix}$$

$$\begin{pmatrix}\xi\\ \eta\end{pmatrix} = \begin{pmatrix}\frac{1}{6} & \frac{1}{6}\\ \frac{5}{6} & -\frac{1}{6}\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}$$

$\xi = \frac{1}{6}(x+y)$, $\eta = \frac{1}{6}(5x-y)$

so general sol'n becomes

$$u = f(\xi) + g(\eta) = f\left(\tfrac{1}{6}(x+y)\right) + g\left(\left(\tfrac{5x-y}{6}\right)\right)$$