A Complete List of Formulas (and Theorems) in MATH400-101, 2018-2019

Part 0:Formula to be given on your final exam paper

1. Charpit's equation for fully nonlinear first order PDE

$$\begin{cases} \frac{dx}{ds} = F_p, x(0) = x_0(\xi) \\ \frac{dy}{ds} = F_q,\ y(0) = y_0(\xi) \\ \frac{dp}{ds} = -F_x - pF_u, p(0) = p_0(\xi) \\ \frac{dq}{ds} = -F_y - qF_u,\ q(0) = q_0(\xi) \\ \frac{du}{ds} = pF_p + qF_q,\ u(0) = u_0(\xi) \end{cases}$$

2. Algebraic equations for negative and positive eigenvalues for Robin boundary conditions: $X'(0) - a_0X(0) = 0, X' + a_lX(l) = 0$

Equation for negative eigenvalues:

$$\lambda = -\gamma^2,\ \tanh(\gamma l) = -\frac{(a_0 + a_l)\gamma}{\gamma^2 + a_0a_l},\ X = \cosh(\gamma x) + \frac{a_0}{\gamma}\sinh(\gamma x)$$

Equation for zero eigenvalue

$$a_0 + a_l + a_0a_ll = 0, \lambda = 0, X = 1 + a_0x$$

Equation for positive eigenvalue

$$\lambda = \beta^2,\ \tan(\beta l) = \frac{(a_0 + a_l)\beta}{\beta^2 - a_0a_l},\ \ X = \cosh(\beta x) + \frac{a_0}{\beta}\sinh(\beta x)$$

3. Diffusion equation with source:

$$u_t = ku_{xx} + f(x,t),$$

$$u(x,0) = \phi$$

$$u(0,t) = h(t), u(l,t) = k(t)$$

Expansion:

$$u = \sum_{n=1}^{\infty} u_n(t)\sin(\frac{n\pi}{l}x)$$

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t)\sin(\frac{n\pi}{l}x)$$

$$\phi(x) = \sum_{n=1}^{\infty} \phi_n\sin(\frac{n\pi}{l}x)$$

Then we need to solve

$$u_n' + k\lambda_nu_n = \frac{2n\pi}{l^2}(h(t) - (-1)^nk(t)) + f_n(t)$$

$$u_n(0) = \phi_n$$

Part I: First Order Equations

$$a(x,y,u)u_x + b(x,y,u)u_y = c(x,y,u)$$

1. Method of Characteristics:

$$\frac{dx}{a} = \frac{dy}{b} = \frac{du}{c}$$

For

$$a(x, y)u_x + b(x, y)u_y = 0$$

the method is as follows: Step 1: solve $\frac{dx}{a} = \frac{dy}{b}$ and find $y = y(x; \xi)$, then solve $\xi = \xi(x, y)$. Stpe 2. $u = u(\xi(x, y))$.

In general case, suppose that the initial conditions are given by $(x_0(\xi), y_0(\xi), u_0(\xi))$. Then we need to solve

$$\begin{cases} \frac{dx}{ds} = a(x, y, u), x(0) = x_0(\xi) \\ \frac{dy}{ds} = b(x, y, u), y(0) = y_0(\xi) \\ \frac{du}{ds} = c(x, y, u), u(0) = u_0(\xi) \end{cases}$$

Find ξ, s in terms of x, y and then put it into the formula for u.

2. General Solutions for

$$a(x, y)u_x + b(x, y)u_y = c(x, y)$$

Method: 1) Solve the characteristics:

$$\frac{dy}{dx} = \frac{b(x, y)}{a(x, y)}$$

to get $F(x, y; \xi) = 0$ and then solve $\xi = f(x, y)$. 2) Change variable

$$x^{'} = x, y^{'} = f(x, y), u(x, y) = U(x^{'}, y^{'})$$

New equation for U:

$$aU_{x'} = c$$

and integrate.

3. Traffic Flow Problem:

$$\rho_t + c(\rho)\rho_x = 0, \rho(x, 0) = \rho_0(x)$$

where $Q^{'}(\rho) = c(\rho)$.

General solution:

$$x = c(\phi(\xi))t + \xi$$

Shock:

$$\frac{ds}{dt} = \frac{Q(\rho_+) - Q(\rho_-)}{\rho_+ - \rho_-}, s(t_0) = x_0$$

Expanding fan:

$$u = H(\Lambda) = H(\frac{x}{t}), \text{ where } c(H(\Lambda)) = \Lambda, \Lambda = \frac{x}{t}$$

4. Fully nonlinear problem:

$$F(x, y, u, u_x, u_y) = 0$$

Charpit's equation:

$$\begin{cases} \frac{dx}{ds} = F_p, x(0) = x_0(\xi) \\ \frac{dy}{ds} = F_q,\ y(0) = y_0(\xi) \\ \frac{dp}{ds} = -F_x - pF_u, p(0) = p_0(\xi) \\ \frac{dq}{ds} = -F_y - qF_u,\ q(0) = q_0(\xi) \\ \frac{du}{ds} = pF_p + qF_q,\ u(0) = u_0(\xi) \end{cases}$$

where (p_0, q_0) is computed via

$$F(x_0, y_0, p_0, q_0) = 0, u_0^{'} = p_0 x_0^{'} + q_0 y_0^{'}$$

Part I: Second order PDEs: general Formula

1. Wave Equation on the whole line:

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t), -\infty < x < +\infty, \\ u(x,0) = \phi(x),\ u_t(x,0) = \psi(x), -\infty < x < +\infty \end{cases} \tag{1}$$

Decomposition of first order operators $\partial_x^2 - \partial_y^2 = (\partial_x - \partial_y)(\partial_x + \partial_y)$ to obtain general solutions $u = f(x-ct) + g(x+ct)$

D'Alembert's formula

$$u(x,t) = \frac{1}{2}(\phi(x-ct) + \phi(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(y)dy + \frac{1}{2c}\int_0^t (\int_{x-c(t-s)}^{x+c(t-s)} f(y,s)dy)ds$$

2. Diffusion Equation on the whole line:

$$\begin{cases} u_t - k u_{xx} = f(x,t), -\infty < x < +\infty, t > 0 \\ u(x,0) = \phi(x), -\infty < x < +\infty \end{cases} \tag{2}$$

Solution formula

$$u(x,t) = \int_{-\infty}^{\infty} S(x-y,t)\phi(y)dy + \int_0^t \int_{-\infty}^{+\infty} S(x-y,t-s)f(y,s)dyds$$

where

$$S(x,t) = \frac{1}{\sqrt{4k\pi t}} e^{-\frac{x^2}{4kt}}$$

3. Wave Equation on the half line:

$$\begin{cases} u_{tt} - c^2 u_{xx} = f(x,t), 0 < x < +\infty, t > 0 \\ u(x,0) = \phi,\ u_t(x,0) = \psi(x), 0 < x < +\infty \\ u(0,t) = 0 \end{cases} \tag{3}$$

Method of Reflection: extend f, ϕ, ψ oddly to $(-\infty, +\infty)$:

$$\phi_{ext} = \begin{cases} \phi(x),\ x > 0; \\ -\phi(-x),\ x < 0 \end{cases} \quad \psi_{ext} = \begin{cases} \psi(x),\ x > 0; \\ -\psi(-x),\ x < 0 \end{cases} \quad f_{ext} = \begin{cases} f(x,t),\ x > 0; \\ -f(-x,t),\ x < 0 \end{cases}$$

$$u(x,t) = \frac{1}{2}(\phi_{ext}(x-ct) + \phi_{ext}(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{ext}(y)dy + \frac{1}{2c}\int_0^t (\int_{x-c(t-s)}^{x+c(t-s)} f_{ext}(y,s)dy)ds$$

There is a similar formula for Neumann boundary condition.

Inhomogeneous BC: $u(0,t) = h(t)$. Use $V(x,t) = u(x,t) - xh(t)$.

4. Heat Equation on the half line:

$$\begin{cases} u_t - c^2 u_{xx} = f(x,t), 0 < x < +\infty, t > 0 \\ u(x,0) = \phi, 0 < x < +\infty \\ u(0,t) = 0 \end{cases} \tag{4}$$

Method of Reflection: extend f, ϕ oddly to $(-\infty, +\infty)$:

$$\phi_{ext} = \begin{cases} \phi(x),\ x > 0; \\ -\phi(-x),\ x < 0 \end{cases} \quad f_{ext} = \begin{cases} f(x,t),\ x > 0; \\ -f(-x,t),\ x < 0 \end{cases}$$

$$u(x,t) = \int S(x-y,t)\phi_{ext}(y)dy + \int_0^t \int S(x-y,t-s)f_{ext}(y,s)dyds$$

There is a similar formula for Neumann boundary condition.
Inhomogeneous BC: $u(0,t) = h(t)$. Use $V(x,t) = u(x,t) - xh(t)$.
5. Wave equation in bounded interval

$$\begin{cases} u_{tt} - c^2 u_{xx} = 0, 0 < x < l, t > 0 \\ u(x,0) = \phi,\ u_t(x,0) = \psi(x), 0 < x < l \\ u(0,t) = u(l,t) \end{cases} \tag{5}$$

Method of Extension: extend f, ϕ, ψ periodically to $(-\infty, +\infty)$:

$$\phi_{ext} = \begin{cases} \phi(x),\ 0 < x < l; \\ -\phi(-x), l < x < 0; \\ \phi(x \pm 2l), \end{cases} \quad \psi_{ext} = \begin{cases} \psi(x),\ 0 < x < l; \\ -\psi(-x), l < x < 0; \\ \psi(x \pm 2l), \end{cases}$$

$$u(x,t) = \frac{1}{2}(\phi_{ext}(x-ct) + \phi_{ext}(x+ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{ext}(y)dy$$

Part III: Boundary Value Problems and Method of Separation of Variables

1. Method of Separation of Variations

Step 1: Find the right separated functions. Plug into PDE and obtain one Eigenvalue Problem (EVP) and one ODE.

Step 2: Solve (EVP) and (ODE)

Step 3: Sum-up. Plug in the inhomogeneous BC and find the coefficients.

2. Standard Eigenvalue Problems

$$X'' + \lambda X = 0, 0 < x < l$$

2.1) Dirichlet BC: $X(0) = X(l) = 0$

$$\lambda_n = \frac{(n\pi)^2}{l^2}, X_n = \sin(\frac{n\pi}{l}x), n = 1, 2, ...$$

2.2) Neumann BC: $X'(0) = X'(l) = 0$

$$\lambda_n = \frac{(n\pi)^2}{l^2}, X_n = \cos(\frac{n\pi}{l}x), n = 0, 1, 2, ...$$

2.3) Periodic BC: $X(0) = X(l), X'(0) = X'(l)$

$$\lambda_n = \frac{(2n\pi)^2}{l^2}, X_n = a\cos(\frac{2n\pi}{l}x) + b\sin(\frac{2n\pi}{l}x), n = 0, 1, 2, ...$$

2.4) Summary of Robin boundary condition eigenvalue problems

$$\begin{cases} X'' + \lambda X = 0, 0 < x < l, \\ X'(0) - a_0 X(0), X'(l) + a_l X(l) = 0 \end{cases}$$

Hyperbola:

$$a_0 + a_l + a_0 a_l l = (a_0 + \frac{1}{l})(a_l + \frac{1}{l}) - \frac{1}{l^2} = 0$$

divides the parameter space (a_0, a_l) into Five Regions. Depending on the regions, the number of negative or zero eigenvalues can be determined.

Equation for negative eigenvalues:

$$\lambda = -\gamma^2,\ \tanh(\gamma l) = -\frac{(a_0 + a_l)\gamma}{\gamma^2 + a_0 a_l},\ X = \cosh(\gamma x) + \frac{a_0}{\gamma}\sinh(\gamma x)$$

Equation for zero eigenvalue

$$a_0 + a_l + a_0 a_l l = 0, \lambda = 0, X_0 = 1 + a_0 x$$

Equation for positive eigenvalue

$$\lambda = \beta^2,\ \tan(\beta l) = \frac{(a_0 + a_l)\beta}{\beta^2 - a_0 a_l},\ \ X = \cosh(\beta x) + \frac{a_0}{\beta}\sinh(\beta x)$$

3. Sturm-Liouville Eigenvalue Problem

3.1. Transforming all second order linear eigenvalue problem to standard form:

$$X^{''} + ax^{'} + bX + \lambda cX = 0$$

$$p = \mu, p^{'} = a\mu, -q = b\mu, w = c\mu$$

$$(p(x)X)^{'} - qw + \lambda w(x)X = 0, 0 < x < l,$$
$$X^{'}(0) - h_0 X(0) = 0, X^{'}(l) + h_1 X(l) = 0$$

Lagrange's identity:

$$\int_0^l [f(pg^{'})^{'} - g(pf^{'})^{'}] = (pfg^{'} - pf^{'}g)|_0^l$$

1) all eigenvalues are real
2) $\lambda_1 > 0$ if $h_0 > 0, h_1 > 0$
3) Different eigenfunctions are orthogonal with respect to the weight function w:

$$\int_0^l w(x) X_n X_m dx = 0$$

4) eigenvalues are discrete and approach to infinity

$$\lambda_1 < \lambda_2 < ... < \lambda_n < ..., \lambda_n \to +\infty$$

5) Expansion with respect to the eigenfunctions:

$$f(x) = \sum_n A_n X_n(x)$$

$$A_n = \frac{\int_0^l X_n f w(x) dx}{\int_0^l X_n^2 w(x) dx}$$

4. Special solvable cases: $aX^{''} + bX^{'} + \lambda X = 0$; $ax^2 X^{''} + bxX^{'} + \lambda X = 0$; Bessel functions of order n:

$$R^{''} + \frac{1}{r}R^{'} - \frac{n^2}{r^2}R + \lambda R = 0, 0 \le r < a,\ R(a) = 0$$

Then

$$\lambda = \mu_{m,n}^2 = \frac{z_{m,n}^2}{a^2}, R(r) = J_n(\frac{z_{m,n}}{a} r), m = 1, 2, ...$$

where Bessel function of order n:

$$J_n^{''} + \frac{1}{r}J_n + J_n - \frac{n^2}{z^2}J_n = 0, J_n(r) \sim r^n$$

The zeroes of J_n is denoted as

$$0 < z_{1,n} < z_{2,n} < ... < z_{m,n} < ...$$

5. Method of Separation of Variables for heat equation/wave equation

1) Diffusion equation without source:

$$u_t = k u_{xx}, 0 < x < l$$
$$u(x, 0) = \phi$$

$$u(0,t) = 0, u(l,t) = 0$$

$$u(x,t) = \sum_{n=1}^{\infty} a_n e^{-k\lambda_n t} \sin(\frac{n\pi}{l}x)$$

where

$$a_n = \frac{2}{l}\int_0^l \phi(x)\sin(\frac{n\pi}{l}x)dx$$

Wave equation without source:

$$u_{tt} = c^2 u_{xx}, 0 < x < l$$

$$u(x,0) = \phi, u_t(x,0) = \psi(x)$$

$$u(0,t) = 0, u(l,t) = 0$$

$$u(x,t) = \sum_{n=1}^{\infty}(a_n \cos(\frac{n\pi c}{l}t) + b_n \sin(\frac{n\pi c}{l}))\sin(\frac{n\pi}{l}x)$$

where

$$a_n = \frac{2}{l}\int_0^l \phi(x)\sin(\frac{n\pi}{l}x)dx$$

$$\frac{n\pi c}{l}b_n = \frac{2}{l}\int_0^l \psi(x)\sin(\frac{n\pi}{l}x)dx$$

Neumann BC: for heat equation

$$u(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-k\lambda_n t}\cos(\frac{n\pi}{l}x)$$

where

$$a_n = \frac{2}{l}\int_0^l \phi(x)\cos(\frac{n\pi}{l}x)dx, n = 0, 1, 2, ...$$

For wave equation:

$$u(x,t) = \frac{a_0 + b_0 t}{2} + \sum_{n=1}^{\infty}(a_n \cos(\frac{n\pi c}{l}t) + b_n \sin(\frac{n\pi c}{l}t))\cos(\frac{n\pi}{l}x)$$

where

$$a_n = \frac{2}{l}\int_0^l \phi(x)\sin(\frac{n\pi}{l}x)dx, n = 0, 1, ...$$

$$b_0 = \frac{2}{l}\int_0^l \psi(x)dx$$

$$\frac{n\pi c}{l}b_n = \frac{2}{l}\int_0^l \psi(x)\sin(\frac{n\pi}{l}x)dx$$

General formula for Sturm-Liouville Eigenvalue Problems: for heat equation

$$u(x,t) = \sum_{n=1}^{\infty} e^{-k\lambda_n t} X_n(x)$$

2) Diffusion equation with source:

$$u_t = ku_{xx} + f(x,t),$$

$$u(x,0) = \phi$$

$$u(0,t) = h(t), u(l,t) = k(t)$$

Expansion:

$$u = \sum_{n=1}^{\infty} u_n(t) \sin(\frac{n\pi}{l} x)$$

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t) \sin(\frac{n\pi}{l} x)$$

$$\phi(x) = \sum_{n=1}^{\infty} \phi_n \sin(\frac{n\pi}{l} x)$$

Then we need to solve

$$u_n^{'} + k\lambda_n u_n = \frac{2kn\pi}{l^2}(h(t) - (-1)^n k(t)) + f_n(t)$$

$$u_n(0) = \phi_n$$

Wave equation with source:

$$u_{tt} = c^2 u_{xx} + f(x,t),$$

$$u(x,0) = \phi, u_t(x,0) = \psi$$

$$u(0,t) = h(t), u(l,t) = k(t)$$

Expansion:

$$u = \sum_{n=1}^{\infty} u_n(t) \sin(\frac{n\pi}{l} x)$$

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t) \sin(\frac{n\pi}{l} x)$$

$$\phi(x) = \sum_{n=1}^{\infty} \phi_n \sin(\frac{n\pi}{l} x)$$

$$\psi(x) = \sum_{n=1}^{\infty} \psi_n \sin(\frac{n\pi}{l} x)$$

Then we need to solve

$$u_n^{''} + c^2\lambda_n u_n = \frac{2n\pi c^2}{l^2}(h(t) - (-1)^n k(t)) + f_n(t)$$

$$u_n(0) = \phi_n, u_n^{'}(0) = \psi_n$$

3) Higher-dimensional Diffusion equation without source:

$$u_t = k(u_{rr} + \frac{1}{r}u_r + \frac{u_{\theta\theta}}{r^2}), 0 \le r < a, 0 \le \theta < 2\pi$$

$$u(r,\theta,0) = \phi(r,\theta)$$

$$u(a,\theta,z) = 0$$

$$u = \sum_m \sum_n e^{-k(\frac{z_{m,n}^2}{a^2})t} J_n(\frac{z_{m,n}}{a} r)(a_{m,n}\cos n\theta + b_{m,n}\sin n\theta)$$

$$a_{m,n} = \frac{\int_0^a \int_0^{2\pi} r\phi(r,z) J_n(\frac{z_{m,n}}{a} r) \cos n\theta d\theta dr}{\int_0^a \int_0^{2\pi} r J_n^2(\frac{z_{m,n}}{a} r) \cos^2(n\theta) d\theta dr}$$

$$\mu_{m,n} = \frac{z_{m,n}}{a}$$

Higher-dimensional Diffusion equation with source:

$$u_t = k(u_{rr} + \frac{1}{r}u_r + \frac{u_{\theta\theta}}{r^2}) + f(r,\theta,t), 0 \le r < a, 0 \le \theta < 2\pi$$

$$u(r,\theta,0) = \phi(r,\theta)$$

$$u(a,\theta,z) = 0$$

$$u = \sum_m \sum_n J_n(\frac{z_{m,n}}{a}r)(a_{m,n}(t)\cos n\theta + b_{m,n}(t)\sin n\theta)$$

$$a'_{m,n} + k\mu^2_{n,m}a_{m,n} = f^1_{n,m}(t), a_{m,n}(0) = \phi^1_{m,n}(0)$$

$$b'_{m,n} + k\mu^2_{n,m}b_{m,n} = f^2_{n,m}(t), a_{m,n}(0) = \phi^2_{m,n}(0)$$

$$\mu_{m,n} = \frac{z_{m,n}}{a}$$

where for example

$$f^1_{m,n} = \frac{\int_0^a \int_0^{2\pi} rf(r,\theta,t)J_n(\frac{z_{m,n}}{a}r)\cos n\theta d\theta dr}{\int_0^a \int_0^{2\pi} rJ_n^2(\frac{z_{m,n}}{a}r)\cos^2(n\theta)d\theta dr}$$

6 Method of separation of variables applied to Laplace equation

6.1) Laplace equation in rectangle and cubes

$$u(x,y) = X(x)Y(y)$$

$$X'' + \lambda_1 X = 0, Y'' + \lambda_2 Y = 0, \lambda_1 + \lambda_2 = 0$$

$$u(x,y,z) = X(x)Y(y)Z(z)$$

6.2) Radial Domains

$$u(r,\theta) = R(r)\Theta(\theta)$$

$$\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{u_{\theta\theta}}{r^2}$$

1) solution to

$$\Delta u = 0, 0 \le r < a, 0 \le \theta < 2\pi, \ u(a,\phi) = h(\phi)$$

is given by

$$u(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{n} r^n(a_n\cos(n\theta) + b_n\sin(n\theta))$$

where

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} h(\phi)d\phi, a_n = \frac{1}{\pi a^n}\int_0^{2\pi} h(\phi\cos(n\phi)d\phi, b_n = \frac{1}{\pi a^n}\int_0^{2\pi} h(\phi\sin(n\phi)d\phi,$$

6.4) Laplace equation on wedges, annulus, exterior of disk

7. Method of separation of variables for Diffusion equation in a disk: polar coordinate

$$u_t = k(u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta})$$

$$u(r,\theta,0) = \phi(r,\theta)$$

$$u(a,\theta,t) = 0$$

The solution is given by

$$u(r,\theta,t) = \sum_{m=1}^{\infty}\sum_{n=0}^{\infty} J_n(\frac{z_{m,n}}{a}r)(a_{m,n}\cos(n\theta) + b_{m,n}\sin(n\theta))e^{-k\frac{z_{m,n}^2}{a^2}t}$$

where

$$a_{m,n} = \frac{\int_0^a \int_0^{2\pi} \phi(r,\theta)\cos(n\theta)d\theta J_n(\frac{z_{m,n}}{a}r)rdr}{\pi\int_0^a J_n^2(\frac{z_{m,n}}{a}r)rdr}$$

$$b_{m,n} = \frac{\int_0^a \int_0^{2\pi} \phi(r,\theta)\sin(n\theta)d\theta J_n(\frac{z_{m,n}}{a}r)rdr}{\pi\int_0^a J_n^2(\frac{z_{m,n}}{a}r)rdr}$$

Part IV: Properties of Second Order PDEs

1. Classification of second order equations

$$a_{11}u_{xx} + 2a_{12}u_{xy} + a_{22}u_{yy} + b_1u_x + b_2u_y = 0$$

Type of PDEs: Elliptic, Parabolic, Hyperbolic

Change of variables to standard form

$$\partial_x = a_{11}\partial_\xi + a_{12}\partial_\eta$$

$$\partial_y = a_{21}\partial_\xi + a_{22}\partial_\eta$$

Then

$$\xi = a_{11}x + a_{21}y$$

$$\eta = a_{12}x + a_{22}y$$

2. Well-posedness of PDE problems: (a) existence (b) uniqueness (c) stability

2.1. Well-posedness of wave equations via d'Alembert's formula

2.2. Well-posedness of heat equation via heat equation formula.

3. For wave equation

$$u_{tt} = c^2u_{xx}$$

Domain of dependence, domain of influence

4. The energy of wave equation

$$E(t) = \frac{1}{2}\int_{-\infty}^{\infty} u_t^2(x,t)dt + \frac{c^2}{2}\int_{-\infty}^{+\infty} u_x^2(x,t)dx$$

$$\frac{dE}{dt} = 0$$

The energy of diffusion equation

$$E(t) = \frac{1}{2}\int_{-\infty}^{+\infty} u^2(x,t)dx$$

$$\frac{dE}{dt} = -k\int_{-\infty}^{\infty} u_x^2 \le 0$$

Uniqueness of wave and diffusion equations by energy method.

5. For Laplace equation

$$\Delta u = f \text{ in } D$$

5.1) Uniqueness: The solution to Dirichlet BC is unique; the solution to Neumann BC is unique, up to a constant; the solution to Robin BC is unique provided $a \ge 0, a \not\equiv 0$.

Uniqueness proved by energy method (and divergence theorem):

$$\int_D u\Delta u = \int_{\partial D} u\frac{\partial u}{\partial \nu} - \int_D |\nabla u|^2$$

5.2) Laplace equation on a disk: Poisson formula

$$\Delta u = 0 \text{ in } 0 \le r < a, 0 \le \theta < 2\pi,\ u(a,\theta) = h(\theta)$$

$$u(r,\theta) = \frac{a^2 - r^2}{2\pi}\int_0^{2\pi} \frac{h(\phi)}{a^2 + r^2 - 2ar\cos(\theta - \phi)}d\phi$$

5.3) Mean Value Theorem: If $\Delta u = 0$ then

$$u(0) = \frac{1}{2\pi}\int_0^{2\pi} u(a,\phi)d\phi$$

5.4) Maximum Principle: If $\Delta u = 0$ in D then $\max_{\bar{D}} u = \max_{\partial D} u, \min_{\bar{D}} u = \min_{\partial D} u$ and equality holds if and only if $u \equiv Constant$