SOLUTION TO MATH 256 ASSIGNMENT 1

- Full mark: 80.
- *Warning:* When f is a trig function, say $\cos(\pi x)$, the Fourier coefficient must be dealt with separately, in this case a_1. This is a common mistake in Q7.

(1) (a) As before, $y = y_h + y_p = C_1\cos(\sqrt{2}x) + C_2\sin(\sqrt{2}x) + \sin x$. Boundary conditions implies $C_1 = C_2 = 0$, so $\boxed{y = \sin x.}$

(b) Here $y_h = C_1\cos x + C_2\sin x$. Let $y_p = x(A\cos x + B\sin x)$ and plugging in we have $y_p'' + y_p = 2(-A\sin x + B\cos x)$. So $A = -\frac{1}{2}$, $B = 0$, $y = C_1\cos x + C_2\sin x - \frac{1}{2}x\cos x$. From $y(0) = 0$, $C_1 = 0$. However, from $y(\pi) = 0$, we get $\frac{\pi}{2} = 0$, which is impossible. $\boxed{\text{Therefore this problem has no solution.}}$

Alternatively, you can multiply both sides of the equation by $\sin x$ and integrate from 0 to π. You will get a contradiction because $\int_0^\pi \sin^2 x\,dx \neq 0$.

(c) As in (b), $y_h = C_1\cos x + C_2\sin x$. Try $y_p = x(A\cos x + B\sin x) + C$ to get $y_p = -\frac{1}{2}x\cos x - \frac{1}{2}$. Then, from $y(0) = 0$, $C_1 = 0$. But $y = C_2\sin x - \frac{1}{2}x\cos x - \frac{1}{2}$ is not consistent with $y(\pi) = 0$. $\boxed{\text{So this problem has no solution.}}$

(d) As in (b), $y_h = C_1\cos x + C_2\sin x$ and $y_p = x(A\cos x + B\sin x)$. This time we get $A = 0$ and $B = \frac{1}{2}$, so $y = C_1\cos x + C_2\sin x + \frac{1}{2}x\sin x$. Now $y(0) = 0$ implies $C_1 = 0$ and we see that $y(\pi) = 0$ is automatically satisfied. Therefore we have a family of solutions $\boxed{y = C_2\sin x + \frac{1}{2}x\sin x.}$

(2) (a) We compute $a_0 = \frac{1}{2\pi}\int_{-\pi}^0 x\,dx = -\frac{\pi}{4}$, $a_k = \frac{1}{\pi}\int_{-\pi}^0 x\cos kx\,dx = \frac{1}{\pi}\left[x\cdot\frac{\sin kx}{k} - 1\cdot\frac{-\cos kx}{k^2}\right]_{-\pi}^0 = \frac{1}{\pi}\frac{1-(-1)^k}{k^2}$, and $b_k = \frac{1}{\pi}\int_{-\pi}^0 x\sin kx\,dx = \frac{(-1)^k}{k}$. So

$$\boxed{f(x) \sim -\frac{\pi}{4} + \sum_{k=1}^\infty \left(\frac{1-(-1)^k}{k\pi}\cos kx + \frac{(-1)^k}{k}\sin kx\right).}$$

(b) Similarly, using $a_0 = \frac{1}{2}\int_{-1}^1 f(x)\,dx$ and $a_j = \int_{-1}^1 f(x)\cos(k\pi x)\,dx$ etc. we have

$$\boxed{f(x) \sim \frac{2}{3} + \sum_{k=1}^\infty \left(\frac{2(-1)^k}{k^2\pi^2}\cos k\pi x + \left(\frac{1-2(-1)^k}{k\pi} - \frac{1-(-1)^k}{k^3\pi^3}\right)\sin k\pi x\right).}$$

(c) We can combine the x integrals which is odd: $a_0 = \frac{1}{4}(\int_{-2}^2 x\,dx + \int_0^2\,dx) = \frac{1}{2}$, $a_k = \frac{1}{2}\int_{-2}^2 x\cos\frac{k\pi x}{2}\,dx + \int_0^2\cos\frac{k\pi x}{2}\,dx = 0$, b_k is computed as usual. These yield

$$\boxed{f(x) \sim \frac{1}{2} + \sum_{k=1}^\infty \frac{2(1-9(-1)^k)}{k\pi}\sin\frac{k\pi x}{2}.}$$

(3) (1)(a) $a_0 = \int_0^1 \sin\pi x\,dx = \frac{2}{\pi}$, $a_1 = 0$ by orthogonality, for $k \geq 2$ using product-to-sum we have

$$\boxed{f(x) \sim \frac{2}{\pi} + \sum_{k=2}^\infty \frac{1-(-1)^{k+1}}{\pi}\left(\frac{1}{k+1} - \frac{1}{k-1}\right)\cos k\pi x.}$$

(1)(b) Since sine is odd the odd extension has Fourier series

$$\boxed{f(x) \sim \sin\pi x.}$$

(2)(a) Again cosine is even so the even extension has Fourier series

$$\boxed{f(x) \sim \cos\pi x.}$$

(2)(b) b_k is computed similarly as the a_k in (1)(a). Hence

$$\boxed{f(x) \sim \sum_{k=2}^{\infty} \frac{1-(-1)^{k+1}}{\pi}\left(\frac{1}{k+1}+\frac{1}{k-1}\right)\sin k\pi x.}$$

(3)(a) By direct computations

$$\boxed{f(x) \sim \frac{3}{2} + \sum_{k=1}^{\infty} \frac{-2(1-(-1)^k)}{k^2\pi^2}\cos k\pi x.}$$

(3)(b)

$$\boxed{f(x) \sim \sum_{k=1}^{\infty} \frac{2(1-2(-1)^k)}{k\pi}\sin k\pi x.}$$

(4) (a) Direct computations yield $\boxed{a_0 = \frac{1}{2},\ a_1 = \frac{2}{\pi^2},\ b_1 = \frac{3}{\pi}.}$

(b) Denote the Fourier series of $f(x)$ by $Sf(x)$. Then we have $\boxed{Sf(-\frac{1}{2}) = -\frac{1}{2},\ Sf(0) = \frac{1}{2}(0+1) = \frac{1}{2},\ Sf(\frac{1}{2}) = 1.}$

(5) Let $u(x,t) = X(x)T(t)$. Then from $\frac{X''}{X} = \frac{T'}{2T} = -\lambda^2$ we first have $X'' + \lambda^2 X = 0$, $X(0) = X(\frac{\pi}{2}) = 0 \Rightarrow X = \sin 2kx$, $\lambda = 2k$, as well as $T' + 2\lambda^2 T = 0 \Rightarrow T = Ce^{-8k^2 t}$. Now a superposition $u = \sum_{k=1}^{\infty} C_k \sin(2kx)e^{-8k^2t}$ matches the initial condition by $C_1 = 1$, $C_4 = 4$, $C_k = 0$ otherwise. Hence

$$\boxed{u(x,t) = \sin(2x)e^{-8t} + 4\sin(8x)e^{-128t}.}$$

(6) We consider a function w linear in x such that $w(0) = 1$ and $w(\pi) = -1$, i.e. $w(x) = 1 - \frac{2}{\pi}x$. Then $u = v + w$, where v satisfies $v_t = 2v_{xx}$, $v(x,0) = 0$, $v(0,t) = 0$, $v(\pi,t) = 0$ for $0 < x < \pi$ and $t > 0$. Let $v(x,t) = X(x)T(t)$. Then $X = \sin(kx)$, $T = Ce^{-2k^2t}$, $v = \sum_{k=1}^{\infty} C_k \sin(kx)e^{-2k^2t}$. The initial condition implies $C_k = 0$ for all k, i.e. $v = 0$. So $\boxed{u(x,t) = 1 - \frac{2}{\pi}x. \text{ As } t \to +\infty,\ u \text{ stays the same as } 1 - \frac{2}{\pi}x.}$

(7) Put $u(x,t) = X(x)T(t)$. Then X solves the Neumann problem on $(0,\pi)$, i.e. the eigenvalues are $k = 0, 1, \ldots$ and $X = \cos(kx)$. Then the superposition gives

$$u(x,t) = A_0 + B_0 t + \sum_{k=1}^{\infty} \cos(kx)(A_k \cos(kt) + B_k \sin(kt)).$$

When $t = 0$, we have $A_2 = 1$ and $A_k = 0$ for any other k. Differentiating and put $t = 0$ we have $B_0 = 1$ and $B_k = 0$ for other k's. Thus

$$\boxed{u(x,t) = t + \cos(2x)\cos(2t).}$$

(8) Put $u(x,y) = X(x)Y(y)$. Taking into account the Dirichlet boundary conditions when $x = 0$ and $x = 1$, we have $\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda^2$, so $\lambda = k\pi$ and $X = \sin(k\pi x)$, $k = 1, 2, \ldots$. The Y-equation $Y'' - (k\pi)^2 Y = 0$ has solution $Y = A\cosh(k\pi y) + B\sinh(k\pi y)$. *(Note that this is preferred over $A_k e^{k\pi y} + B_k e^{-k\pi y}$ because it is simpler when you put $y = 0$.)* Now the superposition

$$u(x,y) = \sum_{k=1}^{\infty} \sin(k\pi x)(A_k \cosh(k\pi y) + B_k \sinh(k\pi y))$$

is to be matched with the boundary conditions on y. On the set $y = 0$, we have

$$\sum_{k=1}^{\infty} A_k \sin(k\pi x) = x.$$

As coefficients of a Fourier sine series,

$$A_k = 2\int_0^1 x \sin(k\pi x)\,dx = 2\left[x\frac{-\cos(k\pi x)}{k\pi} - \frac{-\sin(k\pi x)}{k^2\pi^2}\right]_0^1 = \frac{2(-1)^{k+1}}{k\pi}.$$

Finally we put $y = 1$ and require that $A_k \cosh(k\pi) + B_k \sinh(k\pi) = 0$. Our final solution is

$$\boxed{u(x, y) = \sum_{k=1}^{\infty} \frac{2(-1)^{k+1}}{k\pi} \sin(k\pi x) \left(\cosh(k\pi y) - \frac{\cosh(k\pi)}{\sinh(k\pi)} \sinh(k\pi y) \right).}$$

Note: An even better way is to use the solution $Y = A' \cosh(k\pi(y-1)) + B' \sinh(k\pi(y-1))$ instead. When we put $y = 1$ we get $A' = 0$, for any k. In this way the final solution reads:

$$\boxed{u(x, y) = \sum_{k=1}^{\infty} \frac{2(-1)^{k}}{k\pi \sinh(k\pi)} \sin(k\pi x) \sinh(k\pi(y-1)).}$$

as seen using the corresponding "compound angle formula" for sinh.