SOLUTION TO MATH 256 ASSIGNMENT 1

- Full mark: 70. 10 points each.
- -3 for each mistake within a question, including conceptual ones like confusing the variables s and t, writing equalities of expressions where one involves s and the other t, writing Dirac for Heaviside, wrong plus or minus sign, etc.

(1) (a) For $s > 0$,

$$F(s) = \int_0^\infty t^2 e^{-st}\, dt = \left[t^2 \frac{e^{-st}}{-s} - 2t \frac{e^{-st}}{(-s)^2} + 2 \frac{e^{-st}}{(-s)^3} \right]_0^\infty \quad \text{so} \quad \boxed{F(s) = \frac{2}{s^3} \text{ for } s > 0.}$$

(b) Using the corresponding formula in the solution of assignment 1, we have for $s > 0$,

$$F(s) = \int_0^\infty \cos(2t) e^{-st}\, dt = \left[\frac{e^{-st}(-s\cos(2t) + 2\sin(2t))}{s^2+4} \right]_0^\infty \quad \text{so} \quad \boxed{F(s) = \frac{s}{s^2+4} \text{ for } s > 0}.$$

(c) For $s > 1$,

$$F(s) = \int_0^\infty t e^t e^{-st}\, dt = \left[t \frac{e^{-(s-1)t}}{1-s} - \frac{e^{-(s-1)t}}{(1-s)^2} \right]_0^\infty \quad \text{so} \quad \boxed{F(s) = \frac{1}{(s-1)^2} \text{ for } s > 1.}$$

(2) (a)

$$F(s) = \int_1^2 t e^{-st}\, dt = \left[t \frac{e^{-st}}{-s} - \frac{e^{-st}}{s^2} \right]_1^2 \quad \text{so} \quad \boxed{F(s) = e^{-s}\frac{1+s}{s^2} - e^{-2s}\frac{1+2s}{s^2} \quad \text{for } s > 0}.$$

(b)

$$F(s) = \int_\pi^{2\pi} -\sin(t) e^{-st}\, dt = \left[\frac{e^{-st}(\cos(t) + s\sin(t))}{s^2+1} \right]_\pi^{2\pi} \quad \text{so} \quad \boxed{F(s) = \frac{e^{-2\pi s} + e^{-\pi s}}{s^2+1}.}$$

(3) (a) $\boxed{\frac{A}{s+1}, \frac{B}{s-1}.}$

(b) $\boxed{\frac{A}{s+1}, \frac{B}{s-1}, \frac{Cs+D}{s^2+1}.}$

(c) Since s^2 are cancelled (and $s > 0$), we have the terms $\boxed{\frac{A}{s-4}, \frac{B_1}{s-2}, \frac{B_2}{(s-2)^2}.}$

(d) $\boxed{\frac{A_1 s + B_1}{s^2+5}, \frac{A_2 s + B_2}{(s^2+5)^2}, \frac{C_1}{s+10}, \frac{C_2}{(s+10)^2}, \frac{C_3}{(s+10)^3}.}$

(4) (a) Let $F(s) = \dfrac{2s^2+5s+1}{s^2(s^2+4)} = \dfrac{A_1}{s} + \dfrac{A_2}{s^2} + \dfrac{Bs+C}{s^2+4}$. Clearing denominators,

$$2s^2 + 5s + 1 = A_1 s(s^2+4) + A_2(s^2+4) + (Bs+C)s^2.$$

Comparing coefficients,

$$\begin{aligned} 4A_2 &= 1 \\ 4A_1 &= 5 \\ A_2 + C &= 2 \\ A_1 + B &= 0. \end{aligned}$$

Hence $\boxed{F(s) = \frac{5}{4}\cdot\frac{1}{s} + \frac{1}{4}\cdot\frac{1}{s^2} - \frac{5}{4}\cdot\frac{s}{s^2+4} + \frac{7}{8}\cdot\frac{2}{s^2+4}}$. Taking inverse Laplace transform,
$\boxed{f(t) = \frac{5}{4} + \frac{1}{4}t - \frac{5}{4}\cos(2t) + \frac{7}{8}\sin(2t)}$.

(b) $F(s) = \frac{8s-22}{(s-3)^2+1} = \frac{8(s-3)+2}{(s-3)^2+1}$, i.e. $\boxed{F(s) = 8\cdot\frac{s-3}{(s-3)^2+1} + 2\cdot\frac{1}{(s-3)^2+1}}$. Taking inverse Laplace transform, $\boxed{f(t) = e^{3t}(8\cos(t)+2\sin(t)) = 2e^{3t}(4\cos(t)+\sin(t))}$.

(5) Using the facts that $\mathcal{L}[y'] = sY(s) - y(0)$ and $\mathcal{L}[y''] = s^2Y(s) - sy(0) - y''(0)$, we have:

(a) $(s^2-s-6)Y(s) = (s-1)y(0) + y'(0)$, $Y(s) = \frac{s-2}{(s-3)(s+2)} = \frac{1}{5}\cdot\frac{1}{s-3} + \frac{4}{5}\cdot\frac{1}{s+2}$, so
$\boxed{y(t) = \frac{1}{5}e^{3t} + \frac{4}{5}e^{-2t}}$.

(b) $(s^2+\frac{3}{4}s)Y(s) = -2(s+\frac{3}{4})-3+\frac{1}{s}$, $Y(s) = -2\cdot\frac{1}{s} + \frac{1}{s+\frac{3}{4}} - 4\left(\frac{\frac{3}{4}}{s^2(s+\frac{3}{4})}\right) = -2\cdot\frac{1}{s} + \frac{1}{s+\frac{3}{4}} - 4\left(\frac{(s+\frac{3}{4})-s}{s^2(s+\frac{3}{4})}\right) =$
$-2\cdot\frac{1}{s} + \frac{1}{s+\frac{3}{4}} - 4\cdot\frac{1}{s^2} + 4\cdot\frac{1}{s(s+\frac{3}{4})} = -2\cdot\frac{1}{s} + \frac{1}{s+\frac{3}{4}} - 4\cdot\frac{1}{s^2} + \frac{16}{9}(\frac{1}{s} - \frac{1}{s+\frac{3}{4}})$. Hence $\boxed{y(t) = -\frac{70}{9} + \frac{4}{3}t + \frac{52}{9}e^{-\frac{3}{4}t}}$.

(c) $(s^2-2s+2)Y(s) = (s-2) + \frac{s}{s^2+1}$,

$$
\begin{aligned}
Y(s) &= \frac{(s-1)-1}{(s-1)^2+1} + \frac{s}{(s^2+1)((s-1)^2+1)} \\
&= \frac{s-1}{(s-1)^2+1} - \frac{1}{(s-1)^2+1} + \frac{1}{5}\cdot\frac{s}{s^2+1} - \frac{2}{5}\cdot\frac{1}{s^2+1} - \frac{1}{5}\cdot\frac{s-1}{(s-1)^2+1} + \frac{3}{5}\cdot\frac{1}{(s-1)^2+1} \\
&= \frac{4}{5}\cdot\frac{s-1}{(s-1)^2+1} - \frac{2}{5}\frac{1}{(s-1)^2+1} + \frac{1}{5}\cdot\frac{s}{s^2+1} - \frac{2}{5}\cdot\frac{1}{s^2+1}
\end{aligned}
$$

so $\boxed{y(t) = \frac{4}{5}e^t\cos(t) - \frac{2}{5}e^t\sin(t) + \frac{1}{5}\cos(t) - \frac{2}{5}\sin(t) = \frac{1}{5}\left(e^t(4\cos(t) - 2\sin(t)) + (\cos(t) - 2\sin(t))\right)}$.

(6) (a) $\boxed{f(t) = H(t-2)(t-2)^2}$, $\boxed{F(s) = e^{-2s}\mathcal{L}[t^2] = e^{-2s}\frac{2}{s^3}}$.

(b) $\boxed{f(t) = H(t-1)((t-1)^2+1)}$, $\boxed{F(s) = e^{-s}\mathcal{L}[t^2+1] = e^{-s}(\frac{2}{s^3} + \frac{1}{s})}$.

(c) $\boxed{f(t) = (H(t-1) - H(t-2)) + 2H(t-2) = H(t-1) + H(t-2)}$, $\boxed{F(s) = e^{-s}\frac{1}{s} + e^{-2s}\frac{1}{s}}$.

(d) $\boxed{f(t) = (H(t) - H(t-1))t + H(t-1) = H(t)t - H(t-1)(t-1)}$, $\boxed{F(s) = \frac{1}{s} - e^{-s}\frac{1}{s}}$.

(7) (a) $\boxed{f(t) = t^2e^t}$.

(b) $F(s) = e^{-s}\frac{1}{(s+2)(s-1)} = \frac{1}{3}e^{-s}\frac{1}{s-1} - \frac{1}{3}e^{-s}\frac{1}{s+2}$, $\boxed{f(t) = \frac{1}{3}H(t-1)e^{t-1} - \frac{1}{3}H(t-1)e^{-2(t-1)}}$.

(c) $F(s) = 2e^{-2s}\frac{1}{(s+2)(s-2)} = \frac{1}{2}e^{-2s}\frac{1}{s-2} - \frac{1}{2}e^{-2s}\frac{1}{s+2}$, $\boxed{f(t) = \frac{1}{2}H(t-2)e^{2(t-2)} - \frac{1}{2}H(t-2)e^{-2(t-2)}}$.

(d) $\boxed{f(t) = H(t-1) + H(t-2) - H(t-3) - H(t-4)}$.