SOLUTION TO MATH 256 ASSIGNMENT 1

- Full mark: 70. 10 points each.
- *Warning 1*: **DO NOT** just copy (or, in exams, memorize) a generic phase portrait and put it as your answer, because the principal directions (of the eigenvectors) are not necessarily of that type (see for example Q4(1)).
- *Warning 2*: A center is neutrally stable, that is, stable (trajectory stays close forever if initial point is near the equilibrium) but not asymptotically stable (tends to the equilibrium as $t \to +\infty$). **NOT** undetermined.
- To :(, it is very dangerous not to know linear algebra or complex numbers. Do learn them well!

(1) When no confusion arises, we denote the given matrix by A.

(1) The eigenvalues solves $r^2 - (5+1)r + (5)(1) - (-1)(3) = 0$, i.e. $r^2 - 6r + 8 = 0$. Then $r = 2$ or $r = 4$. The kernel of $A - 2I = \begin{pmatrix} 3 & -1 \\ 3 & -1 \end{pmatrix}$ is spanned by $\begin{pmatrix} 1 \\ 3 \end{pmatrix}$, and that of $A - 4I = \begin{pmatrix} 1 & -1 \\ 3 & -3 \end{pmatrix}$ by $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Hence the eigenpairs are given by $\boxed{r_1 = 2,\ v_1 = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \text{ and } r_2 = 2,\ v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}}$. Similarly we have

(2) $r^2 + 1 = 0$, $\boxed{r_\pm = \pm i}$, $A - (\pm i)I = \begin{pmatrix} 2 \mp i & -1 \\ 5 & -2 \mp i \end{pmatrix}$, $\boxed{v_\pm = \begin{pmatrix} 1 \\ 2 \mp i \end{pmatrix}}$.

(3) The characteristic equation is $-r(r^2 - 1) - (-r - 1) + (1 + r) = 0$, i.e. $(r + 1)^2(r - 2) = 0$.

For the simple eigenvalue $\boxed{r_1 = 2}$, $A - 2I = \begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$. Since there is at most one associated eigenvector for a simple eigenvalue, it is easy to see that the eigenspace is spanned by $\boxed{v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}$.

For the repeated eigenvalue $\boxed{r_2 = -1}$, we consider $A - (-1)I = A + I = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$, which is reduced to $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Its null space is two-dimensional and is spanned by $v_{2,1} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$ and $v_{2,2} = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$.

Note: Look at the dimension of the eigenspace before computing any eigenvector!

(4) Here the characteristic equation is $(r - 2)^3 = 0$, so the only eigenvalue is $\boxed{r_1 = 2}$. Now $A - 2I = \begin{pmatrix} -1 & 1 & 1 \\ 2 & -1 & -1 \\ -3 & 2 & 2 \end{pmatrix}$ is reduced by row operations to $\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. (Indeed, add a multiple of the first row to the others and use the resulting second row to eliminate the non-zero entries in the first column.) Hence the eigenspace is one-dimensional, spanned by $\boxed{v_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}}$.

Warning: **DO NOT** write $v_1 = v_2 = v_3 = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}$ because eigenvectors are by definition linearly independent!

(2) (1) $\text{trace}(A) = e^t + 1 - e^t = 1$. So $\boxed{W = Ce^t}$.

(2) Since we are considering regular ODE, we are consider subintervals of $(-\infty, 0)$ or $(0, +\infty)$ (excluding

0!). A is the coefficient of $\mathbf{x}$ when the left hand side is $\mathbf{x}'$, hence is to be divided by t. So trace$(A) = \frac{1}{t}$ and $\boxed{W = Ce^{\log|t|} = Ct}$, the sign according to the absolute value being absorbed by the constant C.

(3) (1) The eigenpairs are $r_1 = 1$, $v_1 = \begin{pmatrix}1\\1\end{pmatrix}$ and $r_2 = 4$, $v_2 = \begin{pmatrix}1\\2\end{pmatrix}$, giving the general solution $\mathbf{x} = C_1\begin{pmatrix}1\\1\end{pmatrix}e^t + C_2\begin{pmatrix}1\\-2\end{pmatrix}e^{4t}$. The initial condition implies $C_1 = 1$ and $C_2 = 0$, so that $\boxed{\mathbf{x} = \begin{pmatrix}1\\1\end{pmatrix}e^t}$.

(2) The eigenpairs are $r_\pm = 2 \pm i$, $v_\pm = \begin{pmatrix}1\\-1\mp i\end{pmatrix}$. Using Euler's formula $e^{it} = \cos t + i\sin t$, we get a complex solution $e^{2t}(\cos t + i\sin t)\begin{pmatrix}1\\-1-i\end{pmatrix} = e^{2t}\left[\begin{pmatrix}\cos t\\-\cos t+\sin t\end{pmatrix} + i\begin{pmatrix}\sin t\\-\cos t-\sin t\end{pmatrix}\right]$. The real and imaginary parts gives linearly independent real solutions and so the general solution is $\mathbf{x} = e^{2t}\left[C_1\begin{pmatrix}\cos t\\-\cos t+\sin t\end{pmatrix} + C_2\begin{pmatrix}\sin t\\-\cos t-\sin t\end{pmatrix}\right]$. By the initial condition, $C_1\begin{pmatrix}1\\-1\end{pmatrix} + C_2\begin{pmatrix}0\\-1\end{pmatrix} = \begin{pmatrix}1\\0\end{pmatrix}$, so $C_1 = 1$, $C_2 = -1$, and $\boxed{\mathbf{x} = e^{2t}\left[\begin{pmatrix}\cos t\\-\cos t+\sin t\end{pmatrix} - \begin{pmatrix}\sin t\\-\cos t-\sin t\end{pmatrix}\right] = e^{2t}\begin{pmatrix}\cos t-\sin t\\2\sin t\end{pmatrix}}$.

(3) The only eigenvalue is $r_1 = 2$ with associated eigenvector $v_1 = \begin{pmatrix}1\\-1\end{pmatrix}$. To proceed, we look for a *generalized eigenvector* given by the equation $(A - r_1I)v_2 = v_1$, i.e. $\begin{pmatrix}-1&-1\\1&1\end{pmatrix}v_2 = \begin{pmatrix}1\\-1\end{pmatrix}$. An (the) obvious choice is $v_2 = \begin{pmatrix}-1\\0\end{pmatrix}$. So the general solution is $\mathbf{x} = e^{2t}\left[C_1\begin{pmatrix}1\\-1\end{pmatrix} + C_2\left(t\begin{pmatrix}1\\-1\end{pmatrix} + \begin{pmatrix}-1\\0\end{pmatrix}\right)\right]$. The initial condition implies $C_1\begin{pmatrix}1\\-1\end{pmatrix} + C_2\begin{pmatrix}-1\\0\end{pmatrix} = \begin{pmatrix}0\\1\end{pmatrix}$, so $C_1 = C_2 = -1$ and $\boxed{\mathbf{x} = e^{2t}\begin{pmatrix}-t\\t+1\end{pmatrix}}$.

(4) (1) The eigenpairs are $r_1 = 3$, $v_1 = \begin{pmatrix}1\\5\end{pmatrix}$ and $r_2 = -1$, $v_2 = \begin{pmatrix}1\\1\end{pmatrix}$, giving the general solution $\boxed{\mathbf{x} = C_1\begin{pmatrix}1\\5\end{pmatrix}e^{3t} + C_2\begin{pmatrix}1\\1\end{pmatrix}e^t}$. Since the eigenvalues have opposite signs, the phase portrait is an $\boxed{\text{unstable saddle}}$.

(2) The eigenpairs are $r_1 = -2$, $v_1 = \begin{pmatrix}1\\1\end{pmatrix}$ and $r_2 = -4$, $v_2 = \begin{pmatrix}1\\-1\end{pmatrix}$, giving the general solution $\boxed{\mathbf{x} = C_1\begin{pmatrix}1\\1\end{pmatrix}e^{-2t} + C_2\begin{pmatrix}1\\-1\end{pmatrix}e^{-4t}}$. Since the eigenvalues are both negative, it's a $\boxed{\text{(proper) stable node}}$.

(3) The eigenpairs are $r_\pm = \pm i$ and $v_\pm = \begin{pmatrix}1\\2\mp i\end{pmatrix}$. The general solution is $\boxed{\mathbf{x} = C_1\begin{pmatrix}\cos t\\2\cos t+\sin t\end{pmatrix} + C_2\begin{pmatrix}\sin t\\-\cos t+2\sin t\end{pmatrix}}$. Because the eigenvalues are purely imaginary, it is a $\boxed{\text{stable center}}$. To determine the direction, look at specific points. For example, at $\mathbf{x} = \begin{pmatrix}1\\0\end{pmatrix}$ we have $\mathbf{x}' = \begin{pmatrix}2\\1\end{pmatrix}$ which points up (in the x_2 component). From this you know that the trajectories go counter-clockwise.

(4) The eigenpairs are $r_\pm = -1 \pm i$ and $v_\pm = \begin{pmatrix}2\pm i\\1\end{pmatrix}$. The general solution is $\boxed{\mathbf{x} = C_1\begin{pmatrix}2\cos t-\sin t\\\cos t\end{pmatrix} + C_2\begin{pmatrix}\cos t+2\sin t\\\sin t\end{pmatrix}}$. Because the eigenvalues have negative real parts, it is a $\boxed{\text{stable spiral}}$. By the same reasoning as above, trajectories spiral to the origin counter-clockwise.

(5) The only eigenpair is $r_1 = 1$ and $v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. An associated eigenvector, which solves $\begin{pmatrix} 2 & -4 \\ 1 & -2 \end{pmatrix} v_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$, is given by $v_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$. The general solution is $\boxed{\mathbf{x} = e^t \left[C_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} + C_2 \left(t \begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right) \right]}$. Since there is a repeated positive eigenvalue, it is an $\boxed{\text{improper unstable node}}$.

(5) These ODEs are of Euler type, i.e. of the form $t\mathbf{x}' = A\mathbf{x}$. Using the substitution $\mathbf{x} = \xi\, t^r$ gives the characteristic equation $r\xi\, t^r = A\xi\, t^r$, i.e. the eigenvalue problem $A\xi = r\xi$.

(1) As usual we get the eigenpairs $r_1 = 2$, $\xi_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $r_2 = 3$, $\xi_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$. Therefore the general solution is $\boxed{\mathbf{x} = C_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} t^2 + C_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} t^3}$.

(2) We have complex eigenvalues $r_\pm = -2 \pm i$ and $\xi_\pm = \begin{pmatrix} 1 \mp i \\ 1 \end{pmatrix}$. Since $t^i = e^{i \log t} = \cos(\log t) + i \sin(\log t)$, a complex solution is

$$t^{-2}(\cos(\log t) + i \sin(\log t)) \begin{pmatrix} 1 - i \\ 1 \end{pmatrix} = t^{-2} \left[\begin{pmatrix} \cos(\log t) + \sin(\log t) \\ \cos(\log t) \end{pmatrix} + i \begin{pmatrix} -\cos(\log t) + \sin(\log t) \\ \sin(\log t) \end{pmatrix} \right].$$

Hence, $\boxed{\mathbf{x} = t^{-2} \left[C_1 \begin{pmatrix} \cos(\log t) + \sin(\log t) \\ \cos(\log t) \end{pmatrix} + C_2 \begin{pmatrix} -\cos(\log t) + \sin(\log t) \\ \sin(\log t) \end{pmatrix} \right]}$.

(6) We assume that $a \neq 0$, otherwise there is no point putting it in the matrix form. We compute $\mathbf{x}' = \begin{pmatrix} y' \\ y'' \end{pmatrix} = \begin{pmatrix} y' \\ -\frac{b}{a} y' - \frac{c}{a} y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{pmatrix} \mathbf{x}$, so $\boxed{A = \begin{pmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{pmatrix}.}$

$\boxed{\text{Now the eigenvalues of } A \text{ satisfy } r(r + \tfrac{b}{a}) + \tfrac{c}{a} = 0, \text{ i.e. } ar^2 + br + c = 0.}$

(7) We have $\rho\rho' = x_1(2x_1 + x_2) + x_2(-x_1 + 2x_2) = 2x_1^2 + 2x_2^2 = 2\rho^2$ so $\boxed{\rho' = 2\rho}$. Also $\rho^2\theta' = -x_2(2x_1 + x_2) + x_1(-x_1 + 2x_2) = -x_1^2 - x_2^2 = -\rho^2$, i.e. $\boxed{\theta' = -1}$. Without solving the equation, we see that as t increases, the angle θ that (x_1, x_2) makes with the x_1-axis decreases uniformly and and the radius ρ increases proportional to its value. Hence the trajectories spiral out of the origin in the clockwise direction.

Note 1: You are asked to find the equation for ρ' and θ', **NOT** to solve them ... (so don't consider them as your final answer)

Note 2: Trajectories don't intersect each other, due to the uniqueness of solutions to ODEs. When drawing a spiral **DO NOT** draw an intersection at the origin.