Practice Problems

1. $p = \frac{2}{t}$, $q = \frac{\sin t}{t^2}$, $y(\frac{\pi}{2}) = 1$

$$\mu = e^{\int \frac{2}{t}} = t^2, \quad \int \mu g = \int \sin t = -\cos t$$

$$y = \frac{1}{\mu}\left(C + \int \mu g\right) = \frac{1}{t^2}(C - \cos t)$$

$$y(\tfrac{\pi}{2}) = 1 \Rightarrow 1 = \frac{1}{(\frac{\pi}{2})^2}(C - 0) \Rightarrow C = \frac{\pi^2}{4}$$

$$y = \frac{1}{t^2}\left(\frac{\pi^2}{4} - \cos t\right)$$

Interval of Existence:

- sol'n : $t \neq 0$
- eqn : $t \neq 0$
- initial Condition: $t_0 = \frac{\pi}{2}$

Interval of Existence is $(0, +\infty)$

2. $p = -\frac{2\cos t}{\sin t}$, $g = \sin^3 t$

$$\mu = e^{\int p} = e^{-\int \frac{2\cos t}{\sin t}} = e^{-2\ln \sin t} = \frac{1}{\sin^2 t}$$

$$\int \mu g = \int \sin t = -\cos t$$

$$y = \frac{1}{\mu}\left(c + \int \mu g\right) = \sin^2 t\,(c - \sin t)$$

$$y(\tfrac{\pi}{2}) = 1 \Rightarrow \quad 1 = c - 1 \Rightarrow c = 2$$

$$y = \sin^2 t\,(2 - \sin t)$$

Interval of existence: $0 < t < \pi$

- eqn: $\sin t \neq 0 \Rightarrow t \neq 0, \pi$
- sol'n: $\forall t$
- initial condition $t_0 = \frac{\pi}{2}$

3. done in class

4. Bernoulli type: $n = 2$ so

$$v = y^{1-n} = y^{-1}$$

$$y' - \frac{\cos t}{\sin t} y = \sin^2 t\, y^2, \quad p = -\frac{\cos t}{\sin t}, \; g = \sin^2 t$$

$$v' + (1-n) p v = (1-n) g$$

$$v' + \frac{\cos t}{\sin t} v = -\sin^2 t$$

$$\mu = e^{\int \frac{\cos t}{\sin t}} = \sin t \qquad \int \mu g = -\int \sin^3 t = \frac{1}{12}(9 \cos t - \cos 3t)$$

$$y = \frac{1}{\sin t}\left(C + \frac{1}{12}(9\cos t - \cos 3t)\right)$$

$$y = \frac{1}{v} = \frac{12\sin t}{9\cos t - \cos 3t + 12C}$$

$y(\frac{\pi}{2}) = 1 \Rightarrow \quad C = 1$

$$y = \frac{12\sin t}{9\cos t - \cos 3t + 12}$$

Interval of Existence : sol'n: $9\cos t - \cos 3t + 12 \neq 0, \ \forall t$

eqn : $\sin t \neq 0, \quad t \neq 0, \pi$

initial : $t_0 = \frac{\pi}{2}$

$$0 < t < \pi$$

5. separable

$(1+2y)y' = 2x$

$y + y^2 = x^2 + C$

$y(2) = -1 \quad \Rightarrow \quad C = -4$

$y + y^2 = x^2 - 4$

$$y = \frac{-1 \pm \sqrt{1 + 4(x^2 - 4)}}{2} = \frac{-1 \pm \sqrt{4x^2 - 15}}{2}$$

Initial condition:

$$y = \frac{-1-\sqrt{4x^2-15}}{2}$$

Interval of existence: • sol'n $\Rightarrow$ $4x^2-15 \geq 0$

• eqn $\Rightarrow$ $y \neq -\frac{1}{2}$ $\Rightarrow$ $4x^2-15 \neq 0$

• initial condition: $x_0 = 2$

So interval of existence: $x > \frac{\sqrt{15}}{2}$

6. separable

$$\frac{3y^2}{2} - 5y = x^3 - e^x + C$$

$y(0)=1 \Rightarrow C = -\frac{5}{2}$

$$3y^2 - 10y + (2e^x + 5 - 2x^3) = 0$$

$$y = \frac{1}{3}\left(5 \pm \sqrt{10+6x^3-6e^x}\right)$$

initial condition: $y = \frac{1}{3}\left(5 - \sqrt{10+6x^3-6e^x}\right)$

Interval of existence: sol'n: $10+6x^3-6e^x > 0$

$\Rightarrow x_1 < x < x_2$

$x_1 \approx -1.10085$

$x_2 \approx 0.692595$

• eqn: $y \neq \frac{5}{3}$

• initial cond: $x_0 = 0$

so Interval of existance

$$-1.10085 < x < 0.692595$$

7. homogeneous

$$v = \frac{y}{x}, \quad f(x,y) = \frac{x^2+3y^2}{2xy}$$

$$xv' + v = f(1,v)$$

$$f(x,y) = \frac{x^2+3(vx)^2}{2x(vx)} = \frac{1+3v^2}{2v}$$

so $$xv' = \frac{1+v^2}{2v}$$

$$\frac{2v}{1+v^2} v' = \frac{1}{x}$$

$$\log(1+v^2) = \log|x| + C$$

$$1+v^2 = C|x| \qquad v = \pm\sqrt{C|x|-1}$$

8. ~~separable~~ homogeneous

$v = \frac{y}{x}$, $f(x,y) = 1 + \frac{3xy + y^2}{x^2}$

$$xv' + v = 1 + 3v + v^2$$

$$\frac{1}{(1+v)^2} v' = \frac{1}{x}$$

$$-\frac{1}{1+v} = \log|x| + C$$

$$v = \frac{1}{C - \log|x|} - 1$$

9. $\frac{dy}{dx} = ry \log \frac{K}{y}$.

$$\frac{1}{y} y' = r \log K - r \log y$$

Let $u = \log y \Rightarrow u' = r\log K - ru$

$$u = \log K + \log \frac{y_0}{K} e^{-rx}$$

$$y = K \exp\left(\log \frac{y_0}{K} e^{-rx}\right)$$

10. Critical pts of $\frac{dy}{dx} = \sin y$ are $\sin y = 0 \Rightarrow y = k\pi$, $k \in \mathbb{Z}$

$$(\sin y)' \Big|_{y=k\pi} = \cos(k\pi) = (-1)^k = \begin{cases} 1, & k \text{ even} \\ -1, & k \text{ odd} \end{cases}$$

So k = even, $k\pi$ is unstable, k = odd, $k\pi$ is stable

11. A set of solutions $\{y_1, y_2\}$ is a set of fundamental solutions of $y''+py'+qy=0$

if all solutions of $y''+py'+qy=0$ are of the form $c_1 y_1 + c_2 y_2$

12. By Abel's Formula: $W'+pW=0$

Here $p = \frac{t+1}{t^2} = \frac{1}{t} + \frac{1}{t^2}$

So $W = e^{-\int(\frac{1}{t}+\frac{1}{t^2})\,dt} = ce^{-\ln t + \frac{1}{t}} = \frac{c}{t}e^{\frac{1}{t}}$

$W(1)=3 \Rightarrow \frac{c}{1}e^1 = 3 \Rightarrow c = 3e^{-1}$

$W = \frac{3e^{-1}}{t} e^{\frac{1}{t}}$

and $W(5) = \frac{3e^{-1}}{5} e^{\frac{1}{5}} = \frac{3}{5} e^{-\frac{2}{5}}$

13. $p = \frac{x}{x^2}$, $W'+pW=0$

$W = e^{-\int p} = e^{-\int \frac{1}{x}} = \frac{c}{x}$

14. (a) $y = c_1 e^{-t} + c_2 e^{-2t}$

$c_1 + c_2 = 1 \quad \} \Rightarrow c_1 = \frac{2}{3}, \ c_2 = \frac{1}{3}$

(b) $r^2+3r+4=0$

$r=-\frac{3}{2}\pm\frac{\sqrt{5}}{2}i$

$\lambda=-\frac{3}{2}, \quad \mu=\frac{\sqrt{5}}{2}$

$y_1=e^{-\frac{3}{2}t}\cos\frac{\sqrt{5}}{2}t, \quad y_2=e^{-\frac{3}{2}t}\sin\frac{\sqrt{5}}{2}t$

$y=c_1 e^{-\frac{3}{2}t}\cos\frac{\sqrt{5}}{2}t + c_2 e^{-\frac{3}{2}t}\sin\frac{\sqrt{5}}{2}t$

$y(0)=0 \Rightarrow c_1=0$

$y'(0)=1 \Rightarrow \frac{\sqrt{5}}{2}c_2=1 \Rightarrow c_2=\frac{2}{\sqrt{5}}$

(c) $r^2+4r+4=0 \qquad r_1=r_2=-2$

$y_1=e^{-2t}, \quad y_2=te^{-2t}$

$y=(c_1+c_2t)e^{-2t}$

$y(0)=1 \Rightarrow c_1=1$

$y'(0)=1 \Rightarrow -2c_2=1 \Rightarrow c_2=-\frac{1}{2}$

(d) $2r^2+3r+6=0$

$r=\frac{-3\pm\sqrt{9-48}}{4}=-\frac{3}{4}\pm\frac{\sqrt{37}}{4}i$

$\lambda=-\frac{3}{4}, \quad \mu=\frac{\sqrt{37}}{4}$

$y_1=e^{-\frac{3}{4}t}\cos\frac{\sqrt{37}}{4}t \qquad y_2=e^{-\frac{3}{4}t}\sin\frac{\sqrt{37}}{4}t$

$y=c_1y_1+c_2y_2$

15 (a) $y_p = t^s (A\cos t + B\sin t) e^t$

Now $y_1 = e^{-t}$, $y_2 = e^{-2t}$

So $s = 0$

$y_p' = (-A\sin t + B\cos t)e^t + (A\cos t + B\sin t)e^t$

$= ((A+B)\cos t + (B-A)\sin t) e^t$

$y_p'' = (-(A+B)\sin t + (B-A)\cos t) e^t$

$+ ((A+B)\cos t + (B-A)\sin t)e^t$

$= (2B\cos t + (-2A)\sin t)e^t$

$y_p'' + 3y_p' + 2y_p = ((2B + 3(A+B) + 2A)\cos t + (-2A + 3(B-A) + 2B)\sin t)$

$= e^t \cos t$

$2B + 3(A+B) + 2A = 1 \quad \Rightarrow \quad 10A = 1$

$-2A + 3(B-A) + 2B = 0 \Rightarrow 5B = 5A \Rightarrow A = B$

$A = \frac{1}{10} = B$

(b) $y_p = t^s (\cancel{A\cos t + B\sin t}\ A + Bt) e^{-t}$

$s \neq 0 \Rightarrow s = 1 \qquad y_p = t(A+Bt)e^{-t}$

(c) Solve $y''+9y=t\,e^{3t}$ —(1)

$y''+9y=3\sin(3t)$ —(2)

separately .

$y_{p,1}=t^{s_1}(A+Bt)e^{3t}$

$y_{p,2}=t^{s_2}(A\cos3t+B\sin3t)$

Now $y_1=\cos3t$, $y_2=\sin3t$

So $s_1=0$, $s_2=1$

(d) $y''+4y'+4y=t^2e^{2t}-\sin t$

Solve $y''+4y'+4y=t^2e^{2t}$

$y''+4y'+4y=-\sin t$ separately

$y''+4y'+4y=0 \Rightarrow r^2+4r+4=0$ $r_1=r_2=-2$

$y_1=e^{-2t}$ $y_2=te^{-2t}$

$y_{p,1}=t^{s_1}(A_0+A_1t+A_2t^2)e^{2t}$ $\Rightarrow y_{p,1}=(A_0+A_1t+A_2t^2)$

$s_1=0$

$y_{p,2}=t^{s_2}(B_1\cos t+B_2\sin t)$ $\Rightarrow y_{p,2}=B_1\cos t+B_2\sin t$

$s_2=0$

For $y_{p,1} \Rightarrow$

$$y_{p,1}' = 2(A_0 + A_1 t + A_2 t^2)e^{2t} + (A_1 + 2A_2 t)e^{2t}$$

$$= \left(2A_0 + A_1 + (2A_1 + 2A_2)t + 2A_2 t^2\right)e^{2t}$$

$$y_{p,1}'' = 2\left(2A_0 + A_1 + (2A_1 + 2A_2)t + 2A_2 t^2\right)e^{2t}$$

$$+ \left(2A_1 + 2A_2 + 4A_2 t\right)e^{2t}$$

$$= \left(4A_0 + 4A_1 + 2A_2 + (4A_1 + 8A_2)t + 4A_2 t^2\right)e^{2t}$$

$$y_{p,1}'' + 4y_{p,1}' + 4y_{p,1}$$

$$= \Big(4A_0 + 4A_1 + 2A_2 + 4(2A_0 + A_1) + 4A_0$$

$$+ \left[(4A_1 + 8A_2) + 4(2A_1 + 2A_2) + 4A_1\right]t$$

$$+ \left[4A_2 + 8A_2 + 4A_2\right]t^2\Big)e^{2t}$$

$$= t^2 e^{2t}$$

So

$$16A_0 + 8A_1 + 2A_2 = 0 \quad \Rightarrow A_0 = -\frac{3}{8}$$

$$16A_1 + 16A_2 = 0 \quad \Rightarrow A_1 = -\frac{1}{16}$$

$$16A_2 = 1 \quad \Rightarrow A_2 = \frac{1}{16}$$

(e) char. eqn $\quad r^2 - 2r + 4 = 0$

$$r = 1 \pm \sqrt{3}\, i$$

$$\lambda = 1, \quad \mu = \sqrt{3}$$

So $\quad y_1 = e^t \cos\sqrt{3}t, \quad y_2 = e^t \sin\sqrt{3}\, t$

$$y_p = t^s \left(A \cos\sqrt{3}t + B \sin\sqrt{3}\, t \right) e^t$$

$$s \neq 0 \quad \Rightarrow \quad s = 1$$

$$y_p = \left(A t \cos\sqrt{3}\, t + B t \sin\sqrt{3}\, t \right) e^t$$

$$y_p'' - 2y_p' + 4y_p = 4\left(A \cos\sqrt{3}t + B \sin\sqrt{3}t \right) e^t$$

$$+ 2\left(-\sqrt{3}A \sin\sqrt{3}\, t + \sqrt{3}B \cos\sqrt{3}\, t \right) e^t$$

$$- 2\left(A \cos\sqrt{3}\, t + B \sin\sqrt{3}\, t \right) e^t$$

$$= \left((2A + \sqrt{3}B) \cos\sqrt{3}t + (2B - \sqrt{3}A) \sin\sqrt{3}t \right) e$$

$$= e^t \sin\sqrt{3}\, t$$

$$\begin{cases} 2A + \sqrt{3}B = 0 \\ 2B - \sqrt{3}A = 1 \end{cases} \Rightarrow \quad \begin{array}{l} A = \\ B = \end{array}$$

16. (a). $y_1 = e^{-2t}, \quad y_2 = te^{-2t}$

$g = t^{-2}e^{-2t}$

Use the method of variation of parameters

$y_p = u_1 y_1 + u_2 y_2$

$$\begin{cases} u_1' y_1 + u_2' y_2 = 0 \\ u_1' y_1' + u_2' y_2' = g \end{cases}$$

$u_1' e^{-2t} + u_2' t e^{-2t} = 0 \quad \Rightarrow \quad u_1' + u_2' t = 0$

$u_1'(-2e^{-2t}) + u_2'(e^{-2t} - 2te^{-2t}) = t^{-2}e^{-2t}$

$\Rightarrow \quad u_2' e^{-2t} = t^{-2} e^{-2t}$

$u_2' = t^{-2} \Rightarrow u_2 = -\frac{1}{t}$

$u_1' = -u_2' t = -t^{-2}t = -t^{-1}$

$u_1 = -\ln t$

So $y = c_1 e^{-2t} + c_2 t e^{-2t} + (-\ln t)e^{-2t} + (-t^{-1}) t e^{-2t}$

(b) $y_1 = \cos 2t, \quad y_2 = \sin 2t$

$g = \sec^2 2t$

$y_p = u_1 y_1 + u_2 y_2$

$$\begin{cases} u_1' \cos 2t + u_2' \sin 2t = 0 \\ u_1'(-2\sin 2t) + u_2'(2\cos 2t) = \sec^2 2t \end{cases}$$

$\Rightarrow \quad u_1' = -\frac{1}{2}\sin 2t \sec^2 2t = -\frac{1}{2}\frac{\sin 2t}{\cos^2 2t}, \quad u_2' = \frac{1}{2}\cos 2t \sec^2 2t$

$$u_1 = -\frac{1}{2}\int \frac{\sin^3 2t}{\cos^2 2t}\, dt$$

$$= +\frac{1}{4}\int \frac{\sin^2 2t}{\cos^2 2t}\, d(\cos 2t)$$

$$= \frac{1}{4}\int \frac{1-u^2}{u^2}\, du$$

$$= -\frac{1}{4u} + \frac{1}{4}u \qquad u = \cos 2t$$

$$u_2 = \frac{1}{2}\int \cos 2t \sec^2 2t$$

$$= \frac{1}{2}\int \frac{\sin^2 2t}{\cos 2t}\, dt$$

$$= \frac{1}{2}\int \frac{1-\cos^2 2t}{\cos 2t}\, dt = \frac{1}{2}\int \frac{1}{\cos 2t} - \frac{1}{2}\int \cos 2t$$

$$= \frac{1}{4}\left(\ln(1+\cos 2t) - \ln(1-\cos 2t)\right) - \frac{1}{4}\sin 2t$$

(c), $\quad y_p = u_1 t^{-2} + u_2 t^{-1}, \quad g = \dfrac{3t^2-1}{t^2}$

$$u_1' t^{-2} + u_2' t^{-1} = 0 \quad \Rightarrow \quad u_1' + u_2' t = 0$$

$$u_1'(-2t^{-3}) + u_2'(-t^{-2}) = \frac{3t^2-1}{t^2} \quad \Rightarrow \quad u_2' t^{-2} = \frac{3t^2-1}{t^2}$$

$$u_2' = 3t^2-1 \quad \Rightarrow \quad u_2 = t^3 - t$$

$$u_1' = -u_2' t = -3t^3 + t \qquad u_1 = -\frac{3}{4}t^4 + \frac{t^2}{2}$$

(d). $y_1 = \cos\frac{t}{2}$, $y_2 = \sin\frac{t}{2}$

$$y_p = u_1 y_1 + u_2 y_2$$

$$\begin{cases} u_1' \cos\frac{t}{2} + u_2' \sin\frac{t}{2} = 0 \\ u_1'\left(-\frac{1}{2}\sin\frac{t}{2}\right) + u_2'\left(\cos\frac{t}{2}\cdot\frac{1}{2}\right) = 2\sec(t/2) \end{cases}$$

$$u_1' = -4\sin\frac{t}{2}\sec\frac{t}{2}$$

$$u_2' = 4\cos\frac{t}{2}\sec\frac{t}{2}$$

$$u_1 = -4\int \frac{\sin\frac{t}{2}}{\cos\frac{t}{2}}\,dt = 8\ln\left|\cos\frac{t}{2}\right|$$

$$u_2 = 4\int \cos\frac{t}{2}\sec\frac{t}{2} = 4t$$

so $y_p = 8\ln\left|\cos\frac{t}{2}\right|\cos\frac{t}{2} + 4t\sin\frac{t}{2}$

17. $y = y_p + c_1 y_1 + c_2 y_2$, $y_1 = \cos t$, $y_2 = \sin t$

$$y_p = u_1 \cos t + u_2 \sin t$$

$$\begin{cases} u_1' \cos t + u_2' \sin t = 0 \\ u_1'(-\sin t) + u_2' \cos t = g(t) \end{cases} \Rightarrow \begin{aligned} u_1' &= -\sin t\, g(t) \\ u_2' &= \cos t\, g(t) \end{aligned}$$

$$u_1' = -\int \sin t\, g(t) = -\int \sin s\, g(s)\, ds$$

$$u_2 = \int \cos s\, g(s)\, ds$$

$$y_p = u_1 \cos t + u_2 \sin t$$

$$= -\int \sin s \cos t\, g(s)\, ds + \int \sin t \cos s\, g(s)\, ds$$

$$= \int \sin(t-s)\, g(s)\, ds$$

$$y_p = \int_{t_0}^{t} \sin(t-s)\, g(s)\, ds + C_1 \cos t + C_2 \sin t$$

$$\left.\begin{aligned} y_p(t_0) = 0 &\Rightarrow C_1 \cos t_0 + C_2 \sin t_0 = 0 \\ y_p'(t_0) = 0 &\Rightarrow -C_1 \sin t_0 + C_2 \cos t_0 = 0 \end{aligned}\right\} \Rightarrow C_1 = C_2 = 0$$

So the solution is

$$y_p = \int_{t_0}^{t} \sin(t-s)\, g(s)\, ds .$$

18 (a) Euler's type

$$2r(r-1) + 3r + 1 = 0 \qquad 2r^2 + r + 1 = 0$$

$$r = \frac{-1 \pm \sqrt{1-8}}{2 \times 2} = -\frac{1}{4} \pm \frac{\sqrt{7}}{4} i, \quad \lambda = -\frac{1}{4}, \quad \mu = \frac{\sqrt{7}}{4}$$

$$y \ldots t^{-\frac{1}{4}} (\sqrt{7} \ldots) \qquad y \ldots t^{-\frac{1}{4}} (\sqrt{7} \ldots t)$$

(b), (c) are Euler's type. we skip them

(d) $y_1 = t^{-\frac{1}{2}} \sin t \qquad p = \frac{t}{t^2} = \frac{1}{t}$

$y_2 = v y_1 \qquad v' = \frac{W}{y_1^2}$

$W = e^{-\int p} = e^{-\int \frac{1}{t}} = \frac{1}{t}$

so $v' = \frac{\frac{1}{t}}{t^{-1} \sin^2 t} = \frac{1}{\sin^2 t}$

$v = \int \frac{1}{\sin^2 t} = -\cot t$

$y_2 = -\cot t \cdot t^{-\frac{1}{2}} \sin t = -t^{-\frac{1}{2}} \cos t$

(e) $y_1 = e^t, \qquad p = \frac{t}{1-t}$

$y_2 = v y_1, \qquad v' = \frac{W}{y_1^2}$

$W = e^{-\int \frac{t}{1-t} dt} = e^{+\int (1 - \frac{1}{1-t}) dt} = (t-1) e^t$

$v' = \frac{(t-1) e^t}{e^{2t}} = (t-1) e^{-t}$

$v' = -t e^{-t}$

so $y_2 = -t e^{-t} e^t = -t$

(f) $y_1 = \sin t^2, \qquad p = -\frac{1}{t}, \qquad y_2 = v y_1$

$$v' = \frac{W}{y_1^2}$$

$$W = e^{-\int p} = e^{\int \frac{1}{t}} = t$$

$$v' = \frac{t}{(\sin t^2)^2}$$

$$v = \int \frac{t}{(\sin t^2)^2} \overset{u=t^2}{=} \frac{1}{2} \int \frac{du}{\sin^2 u} = -\frac{1}{2} \cot t^2$$

$$y_2 = -\frac{1}{2} \cot t^2 \sin t^2 = -\frac{1}{2} \cos t^2$$

(g) done in class

19. (a). 1) Get y_2. $y_1 = 1+t$, $p = -\frac{1+t}{t}$

$y_2 = v y_1$, $v' = \frac{W}{y_1^2}$

$$W = e^{-\int p} = e^{\int \frac{1+t}{t}} = t e^t$$

$$v' = \frac{te^t}{(1+t)^2}$$

$$v = \frac{e^t}{1+t}$$

$$y_2 = v y_1 = e^t$$

2) $y'' - \frac{1+t}{t} y' + \frac{1}{t} y = t e^{2t}$

$$y_p = u_1 y_1 + u_2 y_2$$

$$\begin{cases} u_1' y_1 + u_2' y_2 = 0 \\ u_1' y_1' + u_2' y_2' = g \end{cases}$$

$$\begin{cases} u_1'(1+t) + u_2' e^t = 0 \\ u_1' + u_2' e^t = t^2 e^{2t} \end{cases}$$

$u_1' t = -t^2 e^{2t}$ $\quad u_1' = -t e^{2t} \Rightarrow u_1 = -\frac{t}{2} e^{2t} + \frac{1}{4} e^{2t}$

$u_2' e^t = t^2 e^{2t} - u_1'$

$u_2' = t^2 e^t + t e^t$

$u_2 = \int t^2 e^t + \int t e^t$

$= t^2 e^t - 2\int t e^t + \int t e^t = t^2 e^t + t e^t - e^t$

(6). 1) $y_1 = e^t$, $\quad p = \frac{t}{1-t}$

$y_2 = v y_1$, $\quad v' = \frac{W}{y_1^2}$

$W = e^{-\int \frac{t}{1-t}} = (t-1) e^t$

$v' = \frac{(t-1)e^t}{e^{2t}}$ $\quad v = \int (t-1) e^{-t} = -t e^{-t}$

$y_2 = -t$. In fact we choose $y_2 = t$

2) $y_p = u_1 y_1 + u_2 y_2$, $\quad g = \frac{2(t-1)^2}{1-t} e^{-t} = -2(t-1) e^{-t}$

$$\Rightarrow \begin{cases} u_1' e^t + u_2' t = 0 \\ u_1' e^t + u_2' = -2(t-1)e^{-t} \end{cases}$$

$$u_2' = -e^{-t} \quad \Rightarrow u_2 = e^{-t}$$

$$u_1' e^t = t e^{-t} \qquad u_1' = t e^{-2t}$$

$$u_1 = -\tfrac{1}{2} t e^{-2t} - \tfrac{1}{4} e^{-2t}$$

(c). 1) $y_1 = t^2$, $\quad p = -\frac{3t}{t^2} = -\frac{3}{t}$

$y_2 = v y_1$, $\quad v' = \frac{W}{y_1^2}$

$$W = e^{-\int p} = t^3$$

$$v' = \frac{t^3}{t^4} = \frac{1}{t}$$

$$v = \ln t$$

$$y_2 = t^2 \ln t$$

2) $y_p = u_1 y_1 + u_2 y_2$, $\quad g = \frac{t^2 \ln t}{t^2} = \ln t$

$$u_1' t^2 + u_2' t^2 \ln t = 0$$

$$u_1'(2t) + u_2'(2t\ln t + t) = \ln t \quad \Rightarrow \quad u_2' = \frac{1}{t} \ln t$$

$$u_2 = \int \frac{1}{t} \ln t = \frac{1}{2} \ln^2 t$$

$$u_1 = -\frac{1}{t} \ln^2 t = -\frac{1}{3} \ln^3 t$$

(d) 1) $y_1 = t^{-\frac{1}{2}}\sin t$, $y_2 = v y_1$, $p = \frac{t}{t^2} = \frac{1}{t}$

$$v' = \frac{W}{y_1^2}$$

$$W = e^{-\int p} = \frac{1}{t}$$

$$v' = \frac{\frac{1}{t}}{t^{-1}\sin^2 t} \qquad v' = \frac{1}{\sin^2 t}$$

$$v = -\cot t$$

$$y_2 = -\cot t \; t^{-\frac{1}{2}}\sin t = -t^{-\frac{1}{2}}\cos t$$

We can take $y_2 = t^{-\frac{1}{2}}\cos t$

2) $y_p = u_1 y_1 + u_2 y_2$, $g = \dfrac{3t^{3/2}\sin t}{t^2} = 3t^{-\frac{1}{2}}\sin t$

$$u_1' t^{-\frac{1}{2}}\sin t + u_2' t^{-\frac{1}{2}}\cos t = 0$$

$$u_1'\left(-\tfrac{1}{2}t^{-\frac{3}{2}}\sin t + t^{-\frac{1}{2}}\cos t\right) + u_2'\left(-\tfrac{1}{2}t^{-\frac{3}{2}}\cos t - t^{-\frac{1}{2}}\sin t\right) = 3t^{-\frac{1}{2}}\sin t$$

$$u_1'\sin t + u_2'\cos t = 0$$

$$u_1'(\cos t) + u_2'(-\sin t) = 3\sin t$$

$$u_1' = 3\sin t\cos t = \tfrac{3}{2}\sin 2t \qquad u_1 = -\tfrac{3}{4}\cos 2t$$

$$u_2' = -3\sin^2 t = -\tfrac{3}{4}(1-\cos 2t), \quad u_2 = -\tfrac{3}{4}t + \tfrac{3}{8}\sin$$