Sol'ns to MATH256-103-2019 Assignment 3

1. (40)

(1) $P=\begin{pmatrix}5 & -1\\ 3 & 1\end{pmatrix}$, $\det(P-rI)=\det\begin{vmatrix}5-r & -1\\ 3 & 1-r\end{vmatrix}=(r-5)(r-1)+3$

$=r^2-6r+8=(r-2)(r-4)$

$r_1=4$, $Pa=ra \Rightarrow \begin{pmatrix}5-4 & -1\\ 3 & 1-4\end{pmatrix}\begin{pmatrix}a_1\\ a_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \Rightarrow a_1-a_2=0 \Rightarrow a^{(1)}=\begin{pmatrix}1\\ 1\end{pmatrix}$

$r_2=2$, $Pa=ra \Rightarrow \begin{pmatrix}5-2 & -1\\ 3 & 1-2\end{pmatrix}\begin{pmatrix}a_1\\ a_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \Rightarrow 3a_1-a_2=0 \Rightarrow a^{(2)}=\begin{pmatrix}1\\ 3\end{pmatrix}$

$X=c_1\begin{pmatrix}1\\ 1\end{pmatrix}e^{4t}+c_2\begin{pmatrix}1\\ 3\end{pmatrix}e^{2t}$

(2) $P=\begin{pmatrix}2 & 1\\ 5 & -2\end{pmatrix}$, $\det(P-rI)=\begin{vmatrix}2-r & 1\\ 5 & -2-r\end{vmatrix}=r^2-4-5=r^2-9=0$

$r_1=3 \Rightarrow \begin{pmatrix}2-3 & 1\\ 5 & -2-3\end{pmatrix}\begin{pmatrix}a_1\\ a_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \Rightarrow -a_1+a_2=0 \Rightarrow a^{(1)}=\begin{pmatrix}1\\ 1\end{pmatrix}$

$r_2=-3 \Rightarrow \begin{pmatrix}2+3 & 1\\ 5 & -2+3\end{pmatrix}\begin{pmatrix}a_1\\ a_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \Rightarrow 5a_1+a_2=0 \Rightarrow a^{(2)}=\begin{pmatrix}1\\ -5\end{pmatrix}$

$X=c_1\begin{pmatrix}1\\ 1\end{pmatrix}e^{3t}+c_2\begin{pmatrix}1\\ -5\end{pmatrix}e^{-3t}$

(3) $P=\begin{pmatrix}2 & -1\\ 5 & -2\end{pmatrix}$, $\det(P-rI)=\begin{vmatrix}2-r & -1\\ 5 & -2-r\end{vmatrix}=r^2-4+5=r^2+1=0 \Rightarrow r=\pm i$

$r_1=i$, $\begin{pmatrix}2-i & -1\\ 5 & -2-i\end{pmatrix}\begin{pmatrix}a_1\\ a_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \Rightarrow (2-i)a_1-a_2=0 \Rightarrow a^{(1)}=\begin{pmatrix}1\\ 2-i\end{pmatrix}$

$\begin{pmatrix}1\\ 2-i\end{pmatrix}e^{it}=\begin{pmatrix}\cos t+i\sin t\\ (2-i)(\cos t+i\sin t)\end{pmatrix}=\begin{pmatrix}\cos t\\ 2\cos t+\sin t\end{pmatrix}+i\begin{pmatrix}\sin t\\ -\cos t+2\sin t\end{pmatrix}$

$X=c_1\begin{pmatrix}\cos t\\ 2\cos t+\sin t\end{pmatrix}+c_2\begin{pmatrix}\sin t\\ -\cos t+2\sin t\end{pmatrix}$

(4) $P=\begin{pmatrix}4 & -2\\ 8 & -4\end{pmatrix}$, $\det(P-rI)=\begin{vmatrix}4-r & -2\\ 8 & -4-r\end{vmatrix}=r^2-16+16=r^2=0$

$r_1=r_2=0$, $e^{rt}=1$, $Pa=ra \Rightarrow \begin{pmatrix}4 & -2\\ 8 & -4\end{pmatrix}\begin{pmatrix}a_1\\ a_2\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix} \Rightarrow 2a_1-a_2=0 \Rightarrow a=\begin{pmatrix}1\\ 2\end{pmatrix}$

$X^{(2)}=ta+\vec{b} \Rightarrow Pb=a \Rightarrow \begin{matrix}4b_1-2b_2=1\\ 8b_1-4b_2=2\end{matrix} \Rightarrow b=\begin{pmatrix}\frac{1}{4}\\ 0\end{pmatrix}$

$X=c_1\begin{pmatrix}1\\ 2\end{pmatrix}+c_2\left(t\begin{pmatrix}1\\ 2\end{pmatrix}+\begin{pmatrix}\frac{1}{4}\\ 0\end{pmatrix}\right)$

2. (1) $P=\begin{pmatrix}2&-1\\-2&3\end{pmatrix}$, $\det(P-rI)=\begin{vmatrix}2-r&-1\\-2&3-r\end{vmatrix}=r^2-5r+4=0$

40

$r_1=1$, $\begin{pmatrix}2-1&-1\\-2&3-1\end{pmatrix}\begin{pmatrix}a_1\\a_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\Rightarrow\begin{pmatrix}a_1\\a_2\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}$

$r_2=4$ $\begin{pmatrix}2-4&-1\\-2&3-4\end{pmatrix}\begin{pmatrix}a_1\\a_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\Rightarrow\begin{pmatrix}a_1\\a_2\end{pmatrix}=\begin{pmatrix}1\\-2\end{pmatrix}$

$x=c_1\begin{pmatrix}1\\1\end{pmatrix}e^{t}+c_2\begin{pmatrix}1\\-2\end{pmatrix}e^{4t}$

$x(0)=\begin{pmatrix}1\\1\end{pmatrix}\Rightarrow c_1\begin{pmatrix}1\\1\end{pmatrix}+c_2\begin{pmatrix}1\\-2\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}\Rightarrow \begin{matrix}c_1+c_2=1\\c_1-2c_2=1\end{matrix}\Rightarrow \begin{matrix}c_1=1\\c_2=0\end{matrix}$

$x=\begin{pmatrix}1\\1\end{pmatrix}e^{t}$

(2) $P=\begin{pmatrix}1&-1\\5&3\end{pmatrix}$, $\det(P-rI)=\begin{vmatrix}1-r&-1\\5&3-r\end{vmatrix}=r^2-4r+8=(r-2)^2+4=0$

$r=2\pm2i\Rightarrow r_1=2+2i$, $\begin{pmatrix}1-2-2i&-1\\5&3-2-2i\end{pmatrix}\begin{pmatrix}a_1\\a_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\Rightarrow a=\begin{pmatrix}1\\-1-2i\end{pmatrix}$

$\begin{pmatrix}1\\-1-2i\end{pmatrix}e^{(2+2i)t}=\begin{pmatrix}e^{2t}\cos2t+ie^{2t}\sin2t\\(-1-2i)(e^{2t}\cos2t+ie^{2t}\sin2t)\end{pmatrix}=\begin{pmatrix}e^{2t}\cos2t\\-e^{2t}\cos2t+2e^{2t}\sin2t\end{pmatrix}+i\begin{pmatrix}e^{2t}\sin2t\\-2e^{2t}\cos2t-e^{2t}\sin2t\end{pmatrix}$

$x=c_1\begin{pmatrix}e^{2t}\cos2t\\-e^{2t}\cos2t+2e^{2t}\sin2t\end{pmatrix}+c_2\begin{pmatrix}e^{2t}\sin2t\\-2e^{2t}\cos2t-e^{2t}\sin2t\end{pmatrix}$

$x(0)=\begin{pmatrix}1\\0\end{pmatrix}\Rightarrow c_1\begin{pmatrix}1\\-1\end{pmatrix}+c_2\begin{pmatrix}0\\-2\end{pmatrix}=\begin{pmatrix}1\\0\end{pmatrix}\Rightarrow \begin{matrix}c_1=1\\c_2=-\frac{1}{2}\end{matrix}$

(3) $P=\begin{pmatrix}1&-1\\1&3\end{pmatrix}$; $\det(P-rI)=\begin{vmatrix}1-r&-1\\1&3-r\end{vmatrix}=r^2-4r+3+1=(r-2)^2$

$r_1=r_2=2$, $\begin{pmatrix}1-2&-1\\1&3-2\end{pmatrix}\begin{pmatrix}a_1\\a_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\Rightarrow a=\begin{pmatrix}1\\-1\end{pmatrix}$

$x^{(1)}=\begin{pmatrix}1\\1\end{pmatrix}e^{2t}$, $x^{(2)}=t\begin{pmatrix}1\\1\end{pmatrix}e^{2t}+be^{2t}$

$Pb=2b+\begin{pmatrix}1\\1\end{pmatrix}\Rightarrow\begin{pmatrix}1-2&-1\\1&3-2\end{pmatrix}\begin{pmatrix}b_1\\b_2\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}\Rightarrow -b_1-b_2=1\Rightarrow b=\begin{pmatrix}0\\-1\end{pmatrix}$

$x^{(2)}=t\begin{pmatrix}1\\1\end{pmatrix}e^{2t}+\begin{pmatrix}0\\-1\end{pmatrix}e^{2t}$

$x=c_1\begin{pmatrix}1\\1\end{pmatrix}e^{2t}+c_2\left(t\begin{pmatrix}1\\1\end{pmatrix}e^{2t}+\begin{pmatrix}0\\-1\end{pmatrix}e^{2t}\right)$

$x(0)=\begin{pmatrix}0\\1\end{pmatrix}\Rightarrow c_1\begin{pmatrix}1\\1\end{pmatrix}+c_2\begin{pmatrix}0\\-1\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}\Rightarrow \begin{matrix}c_1=0\\c_2=-1\end{matrix}$

$x=-\left(t\begin{pmatrix}1\\1\end{pmatrix}e^{2t}+\begin{pmatrix}0\\-1\end{pmatrix}e^{2t}\right)$

(4) $P=\begin{pmatrix}3&-4\\1&-1\end{pmatrix}$, $\det(P-rI)=\begin{vmatrix}3-r&-4\\1&-1-r\end{vmatrix}=r^2-2r-3+4=(r-1)^2$

$r_1=r_2=1$, $\begin{pmatrix}3-1&-4\\1&-1-1\end{pmatrix}\begin{pmatrix}a_1\\a_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\Rightarrow\begin{pmatrix}a_1\\a_2\end{pmatrix}=\begin{pmatrix}2\\1\end{pmatrix}$

$x^{(1)}=\begin{pmatrix}2\\1\end{pmatrix}e^t$, $x^{(2)}=t\begin{pmatrix}2\\1\end{pmatrix}e^t+be^t$

$Pb=b+\begin{pmatrix}2\\1\end{pmatrix}\Rightarrow\begin{pmatrix}3-1&-4\\1&-1-1\end{pmatrix}\begin{pmatrix}b_1\\b_2\end{pmatrix}=\begin{pmatrix}2\\1\end{pmatrix}\Rightarrow$ $b_1-2b_2=1$, $b=\begin{pmatrix}1\\0\end{pmatrix}$

$x=c_1\begin{pmatrix}2\\1\end{pmatrix}e^t+c_2\left(t\begin{pmatrix}2\\1\end{pmatrix}e^t+\begin{pmatrix}1\\0\end{pmatrix}e^t\right)$

$x(0)=\begin{pmatrix}0\\1\end{pmatrix}\Rightarrow c_1\begin{pmatrix}2\\1\end{pmatrix}+c_2\begin{pmatrix}1\\0\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}\Rightarrow$ $2c_1+c_2=0\Rightarrow c_2=-2$, $c_1=1$

$x=\begin{pmatrix}2\\1\end{pmatrix}e^t-2\left(t\begin{pmatrix}2\\1\end{pmatrix}e^t+\begin{pmatrix}1\\0\end{pmatrix}e^t\right)$

3. 40

(1) (a) $P=\begin{pmatrix}-2&1\\-5&4\end{pmatrix}$, $\det(P-rI)=\begin{vmatrix}-2-r&1\\-5&4-r\end{vmatrix}=(r+2)(r-4)+5=r^2-2r-3=0$

$r_1=-1$, $r_2=3$

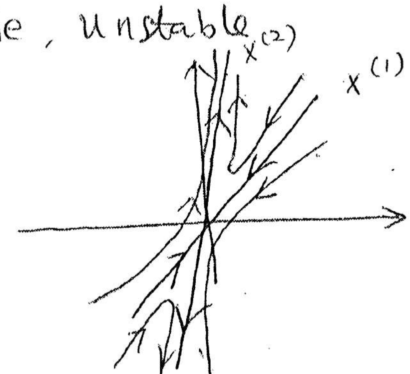
$r_1=-1\Rightarrow\begin{pmatrix}-2-r_1&1\\-5&4-r_1\end{pmatrix}\begin{pmatrix}a_1\\a_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\Rightarrow\begin{pmatrix}a_1\\a_2\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}$

$r_2=3\Rightarrow\begin{pmatrix}-2-3&1\\-5&4-3\end{pmatrix}\begin{pmatrix}a_1\\a_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\Rightarrow\begin{pmatrix}a_1\\a_2\end{pmatrix}=\begin{pmatrix}1\\5\end{pmatrix}$

$x=c_1\begin{pmatrix}1\\1\end{pmatrix}e^{-t}+c_2\begin{pmatrix}1\\5\end{pmatrix}e^{3t}$

(b). saddle, unstable

(c)



(2) (a) $P=\begin{pmatrix}-3&1\\1&-3\end{pmatrix}$, $\det(P-rI)=\begin{vmatrix}-3-r&1\\1&-3-r\end{vmatrix}$
$=(r+3)^2-1=0$

$r_1=-2$, $r_2=-4$

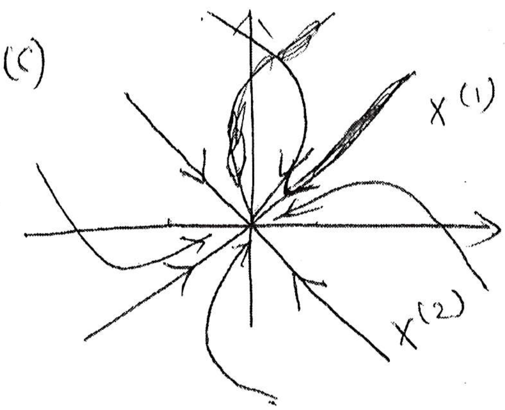
$r_1=-2$, $\begin{pmatrix}-3+2&1\\1&-3+2\end{pmatrix}\begin{pmatrix}a_1\\a_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\Rightarrow\begin{pmatrix}a_1\\a_2\end{pmatrix}=\begin{pmatrix}1\\1\end{pmatrix}$

$r_2=-4$ $\begin{pmatrix}-3+4&1\\1&-3+4\end{pmatrix}\begin{pmatrix}a_1\\a_2\end{pmatrix}=\begin{pmatrix}0\\0\end{pmatrix}\Rightarrow\begin{pmatrix}a_1\\a_2\end{pmatrix}=\begin{pmatrix}1\\-1\end{pmatrix}$

$x=c_1\begin{pmatrix}1\\1\end{pmatrix}e^{-2t}+c_2\begin{pmatrix}1\\-1\end{pmatrix}e^{-4t}$

(b) node, unstable

(c)



(3) (a) $P = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}$, $\det(P - rI) = \begin{vmatrix} 2-r & -5 \\ 1 & -2-r \end{vmatrix}$

$= r^2 - 4 + 5 = r^2 + 1 = 0$

$r_1 = i, \ r_2 = -i$

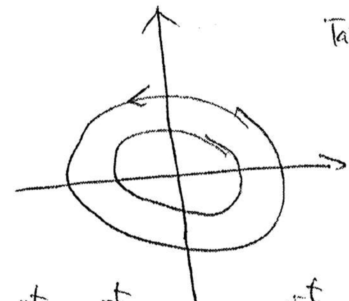
$\begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}$

$\begin{pmatrix} 2+i \\ 1 \end{pmatrix} e^{it} = \begin{pmatrix} (2+i)(\cos t + i \sin t) \\ \cos t + i \sin t \end{pmatrix} = \begin{pmatrix} 2\cos t - \sin t \\ \cos t \end{pmatrix}$

$+ i \begin{pmatrix} 2\sin t + \cos t \\ \sin t \end{pmatrix}$

(b) center, stability undetermined

(c)



Take $\begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow X' = \begin{pmatrix} -5 \\ -2 \end{pmatrix}$

counter-clockwise

(4) (a) $P = \begin{pmatrix} 1 & -5 \\ 1 & -3 \end{pmatrix}$

$\det(P - rI) = \begin{vmatrix} 1-r & -5 \\ 1 & -3-r \end{vmatrix} = (r+3)(r-1) + 5$

$= r^2 + 2r + 2$

$= (r+1)^2 + 1$

$r_1 = -1 + i, \quad \begin{pmatrix} 2-i & -5 \\ 1 & -2-i \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

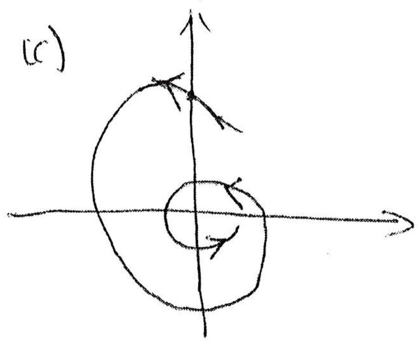
$\Rightarrow \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 2+i \\ 1 \end{pmatrix}$

$\begin{pmatrix} 2+i \\ 1 \end{pmatrix} e^{-t+it} = \begin{pmatrix} (2+i)(e^{-t}\cos t + i e^{-t} \sin t) \\ e^{-t}\cos t + i e^{-t} \sin t \end{pmatrix} = \begin{pmatrix} 2e^{-t}\cos t - e^{-t}\sin t \\ e^{-t}\cos t \end{pmatrix} + i \begin{pmatrix} e^{-t}\cos t + 2e^{-t}\sin t \\ e^{-t}\sin t \end{pmatrix}$

$X = c_1 \begin{pmatrix} 2e^{-t}\cos t - e^{-t}\sin t \\ e^{-t}\cos t \end{pmatrix} + c_2 \begin{pmatrix} e^{-t}\cos t + 2e^{-t}\sin t \\ e^{-t}\sin t \end{pmatrix}$

(b) Spiral, stable

(c)



at $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $X' = \begin{pmatrix} -5 \\ -3 \end{pmatrix}$

(5) $P = \begin{pmatrix} 3 & -4 \\ 1 & -1 \end{pmatrix}$, $\det(P - rI) = \begin{pmatrix} 3-r & -4 \\ 1 & -1-r \end{pmatrix}$

$= r^2 - 2r + 1 = 0$

$r_1 = r_2 = 1, \quad \begin{pmatrix} 3-1 & -4 \\ 1 & -1-1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\Rightarrow \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

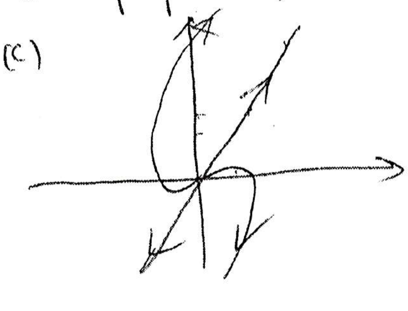
$X^{(2)} = t\begin{pmatrix} 2 \\ 1 \end{pmatrix} e^t + b e^t$

$Pb = b + \begin{pmatrix} 2 \\ 1 \end{pmatrix} \Rightarrow b_1 - 2b_2 = 1 \Rightarrow b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

$X = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^t + c_2 \left(t\begin{pmatrix} 2 \\ 1 \end{pmatrix} e^t + \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^t \right)$

(b) Impropernode, unstable

(c)



4. 20

(1) $x = t^r \vec{a} \Rightarrow x' = r t^{r-1} \vec{a} \Rightarrow t x' = t^r r \vec{a} = t^r P a$

$\Rightarrow Pa = r a \Rightarrow \begin{vmatrix} 1-r & 1 \\ -2 & 4-r \end{vmatrix} = 0 \Rightarrow r^2 - 5r + 6 = 0$

$\Rightarrow r_1 = 2,\ r_2 = 3$

$\begin{pmatrix} 1-r_1 & 1 \\ -2 & 4-r_1 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\begin{pmatrix} 1-r_2 & 1 \\ -2 & 4-r_2 \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

$x = c_1 t^2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 t^3 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

(2)

(2) $\begin{pmatrix} -3-r & -2 \\ 1 & -1-r \end{pmatrix} = 0 \Rightarrow (r+3)(r+1) + 2 = 0$

$r^2 + 4r + 5 = 0 \Rightarrow r_1 = -2+i,\ r_2 = -2-i$

$\begin{pmatrix} -3+2-i & -2 \\ 1 & -1+2-i \end{pmatrix}\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \begin{pmatrix} 1+i \\ 1 \end{pmatrix}$

$\begin{pmatrix} 1+i \\ 1 \end{pmatrix} t^{-2+i} = \begin{pmatrix} (t^{-2}\cos(\ln t) + i t^{-2}\sin(\ln t))(1+i) \\ t^{-2}\cos(\ln t) + i t^{-2}\sin(\ln t) \end{pmatrix}$

$X = c_1 \begin{pmatrix} t^{-2}\cos\ln t - t^{-2}\sin(\ln t) \\ t^{-2}\cos(\ln t \end{pmatrix} + c_2 \begin{pmatrix} t^{-2}\cos\ln t + t^{-2}\sin\ln t \\ t^{-2}\sin(\ln t) \end{pmatrix}$

5. 20 Let $x_1 = y,\ x_2 = y' \Rightarrow$

$x_1' = x_2$

$x_2' = y'' = -\frac{b}{a}y' - \frac{c}{a}y = -\frac{c}{a}x_1 - \frac{b}{a}x_2$

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -\frac{c}{a} & -\frac{b}{a} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\det(P - rI) = \begin{vmatrix} 0-r & 1 \\ -\frac{c}{a} & -\frac{b}{a} - r \end{vmatrix} = r\left(r + \frac{b}{a} r\right) + \frac{c}{a} = \frac{1}{a}\left(ar^2 + br + c\right) = 0$$

$$x' = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} x \Rightarrow \begin{cases} x_1' = -x_1 - x_2 \\ x_2' = x_1 - x_2 \end{cases}$$

6.
$$\rho\rho' = x_1 x_1' + x_2 x_2'$$
$$= x_1(-x_1 - x_2) + x_2(x_1 - x_2)$$
$$= -x_1^2 - x_2^2 = -\rho^2$$
$$\Rightarrow \rho' = -\rho$$

$$\theta' = \frac{-x_2 x_1' + x_1 x_2'}{x_1^2 + x_2^2} = \frac{-x_2(-x_1 - x_2) + x_1(x_1 - x_2)}{x_1^2 + x_2^2} = \frac{x_1^2 + x_2^2}{x_1^2 + x_2^2} = 1$$

Hence $\rho' = -\rho$, $\theta' = 1$.

Now let $X(0) = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix}$; $\rho_0 = \sqrt{x_1^2(0) + x_2^2(0)}$, $\theta_0 = \arctan\left(\frac{x_2(0)}{x_1(0)}\right)$. Then

$$\rho = \rho_0 e^{-t}, \quad \theta = \theta_0 + t$$